Momentum Deficit due to Dissipation (Momentum Bath?)

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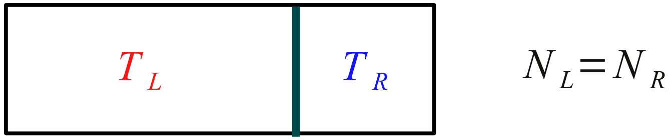
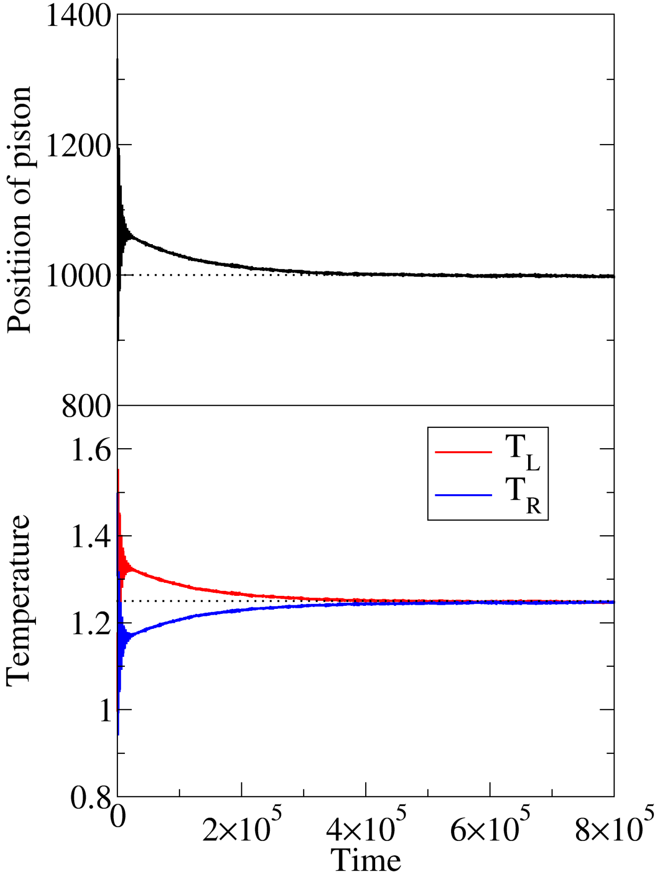
Constructive Fluctuations in Small Devices
(San Diego, 7/8-7/9, 2013)

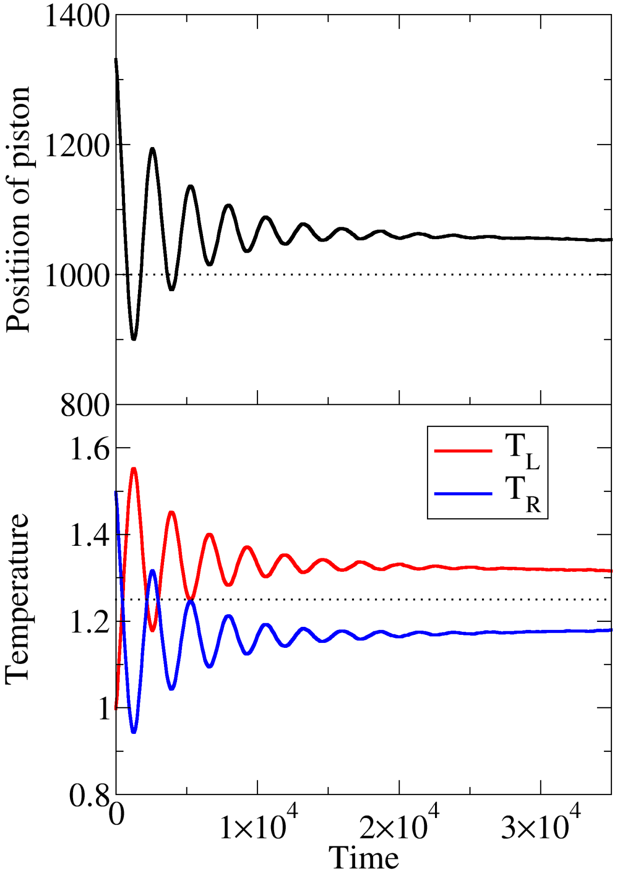
Contents

- Motivation – Examples where the linear Langevin theory fails.
- The Langevin equation can't see the symmetry of a Brownian object in thermal equilibrium. That is OK.
- Not OK in non-equilibrium situations.
- Momentum Deficit due to Dissipation (MDD) is introduced.
- MDD explains the missing force.
- Sasa's paradox and momentum bath.

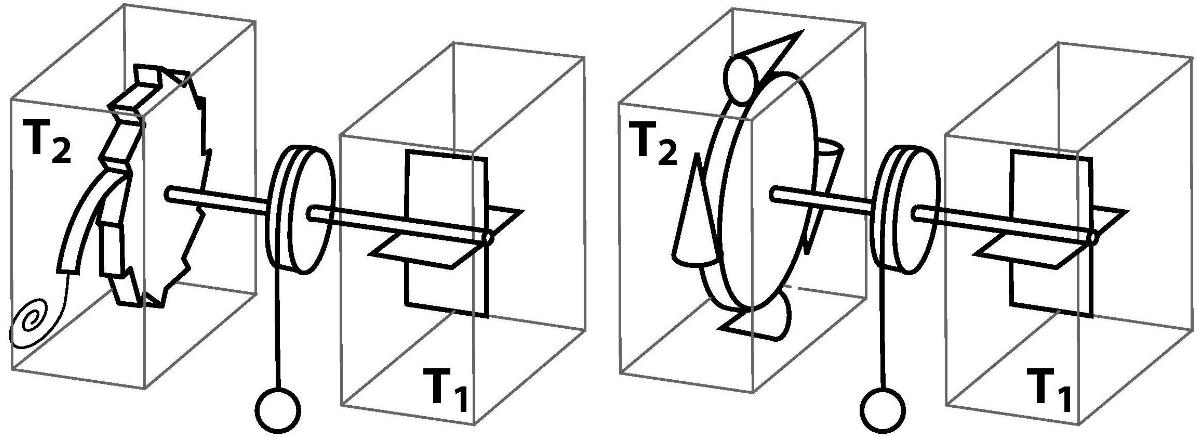
Adiabatic Piston

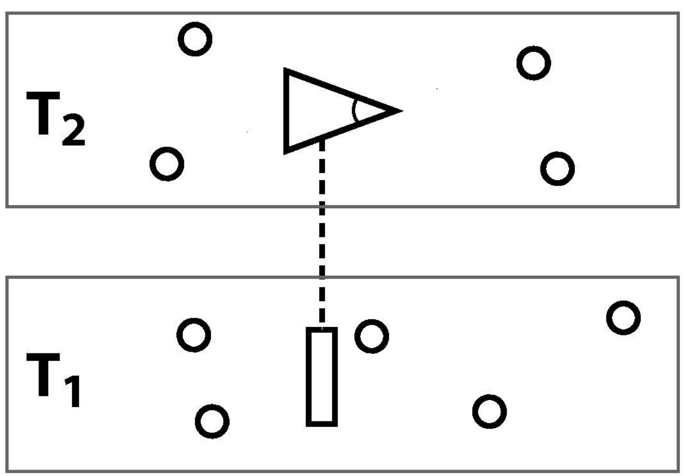


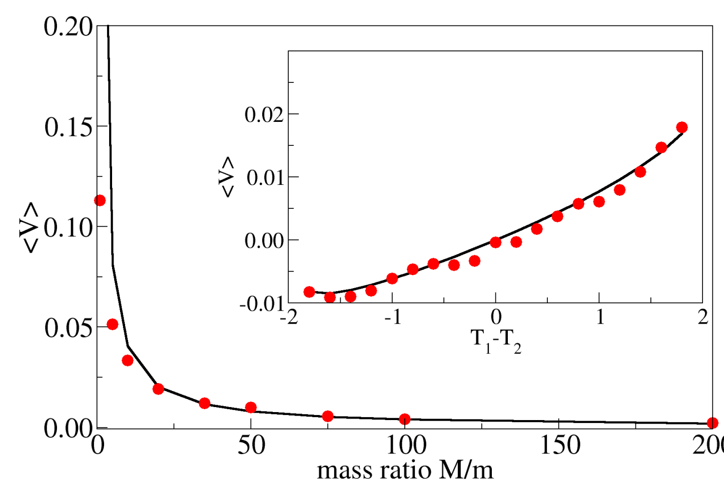




Feynman Ratchet







Van den Broeck, R.K. and Meurs, PRL **93** (2004) 090601

What is the force on a Brownian object exerted by an *environment*?

If the environment is an ideal heat bath (energy reservoir at equilibrium)

$$M\dot{V} = -\gamma V + \sqrt{2\gamma k_B T}\,\xi - \nabla U(X)$$

$$\langle \xi(t) \rangle = 0$$

$$M\langle \dot{V} \rangle = -\gamma \langle V \rangle - \langle \nabla U(X) \rangle$$

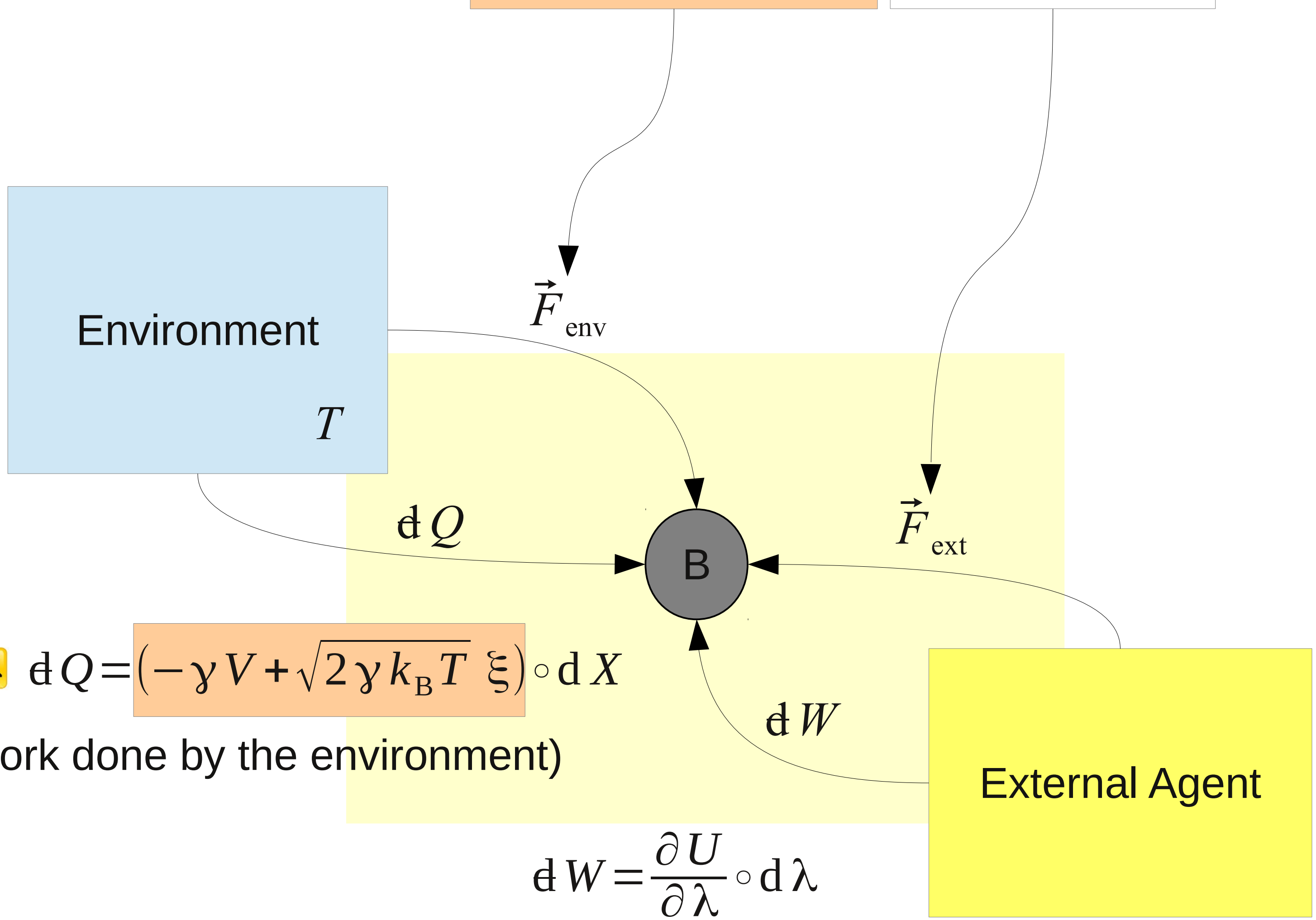
$$\langle \xi(t)\xi(t') \rangle = \delta(t-t')$$

What is the force on a Brownian object exerted by the environment under a non-equilibrium condition?

Is the usual heat bath good enough?
If not, is there a universal environment with *desired properties*?

Stochastic Thermodynamics: Driven Non-equilibrium Processes

$$M\dot{V} = -\nabla U(X) \; \boxed{-\gamma V + \sqrt{2\gamma k_B T}\,\xi} \; \boxed{-\nabla U_{ext}(X,\lambda)}$$



$$đQ = \left(-\gamma V + \sqrt{2\gamma k_B T}\,\xi\right) \circ \mathrm{d}X$$

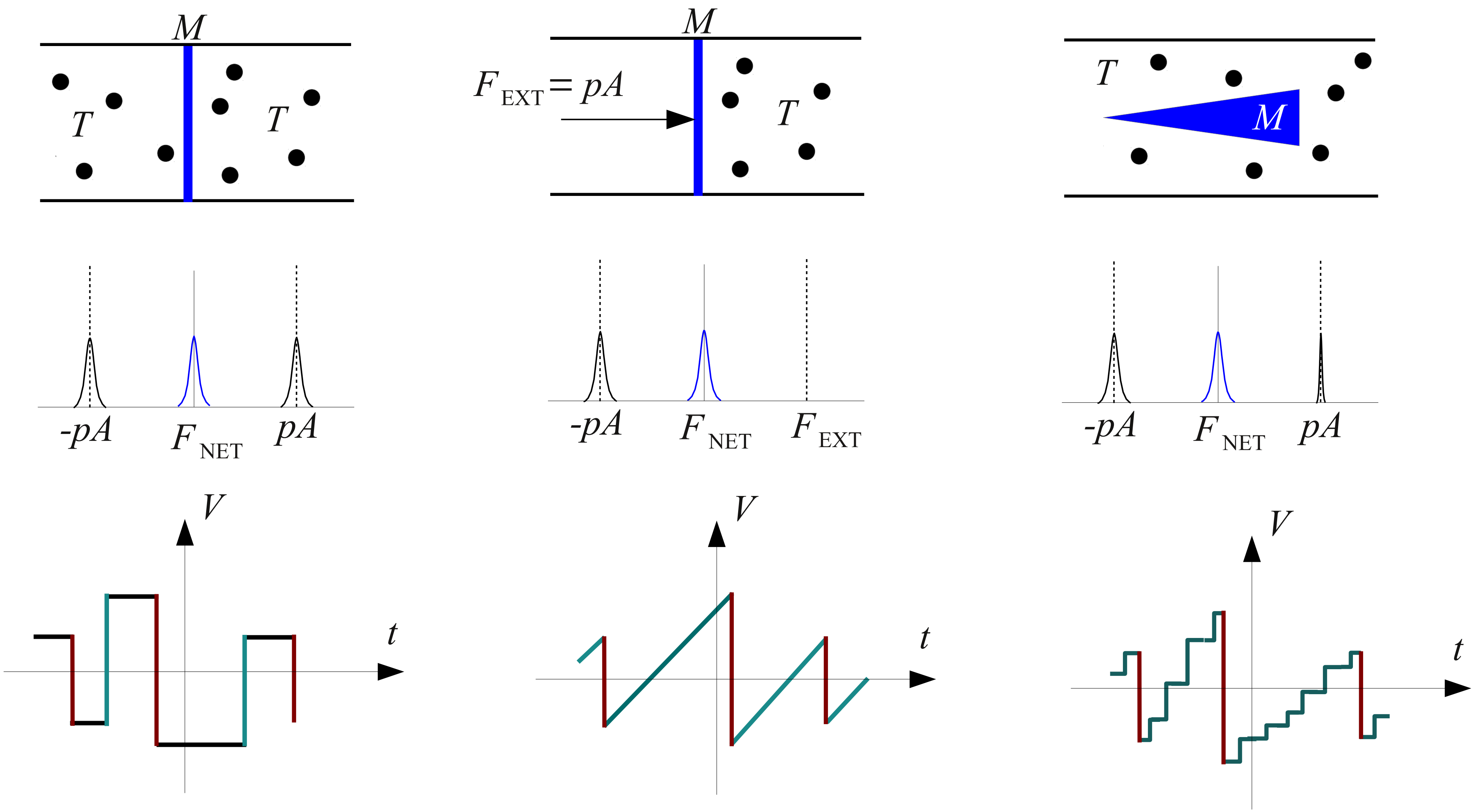
(work done by the environment)

$$đW = \frac{\partial U}{\partial \lambda} \circ \mathrm{d}\lambda$$

Sekimoto, *Stochastic Energetics* (Springer, 2010).

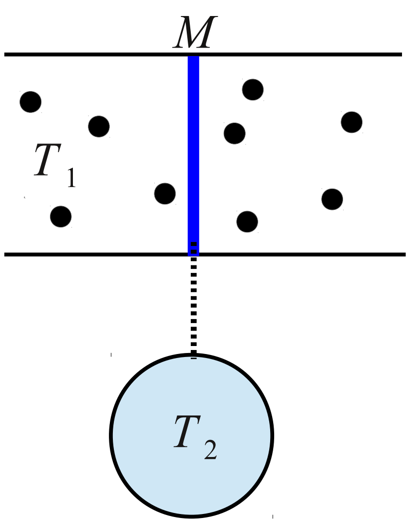
Lagevin Eq. can't see asymmetric situation around the Brownian object.
(Coarse graining hides it.)

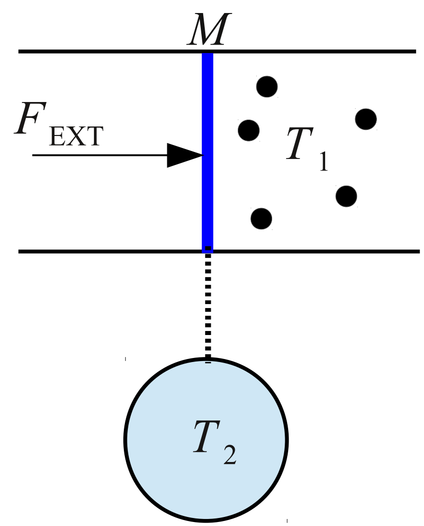
$$M\dot{V} = -\gamma V + \sqrt{2\gamma k_B T}\,\xi + F_{EXT} \qquad \langle V \rangle = 0, \quad \langle V^2 \rangle = \frac{k_B T}{M}$$

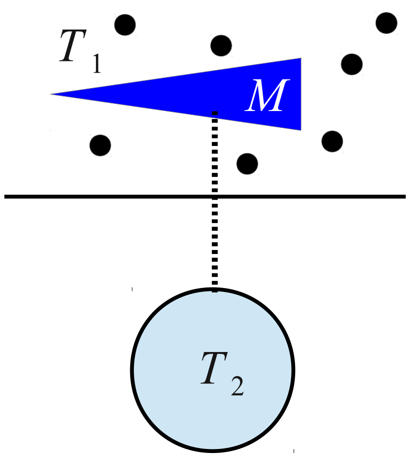


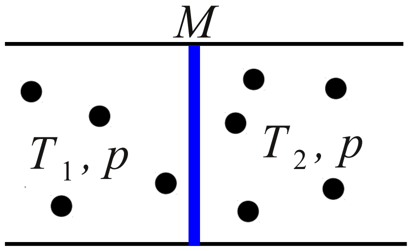
Non-equilibrium steady states illuminate the asymmetry.

$$M\dot{V} = -\gamma_1 V + \sqrt{2\gamma_1 k_B T_1}\,\xi_1 + F_{EXT}$$



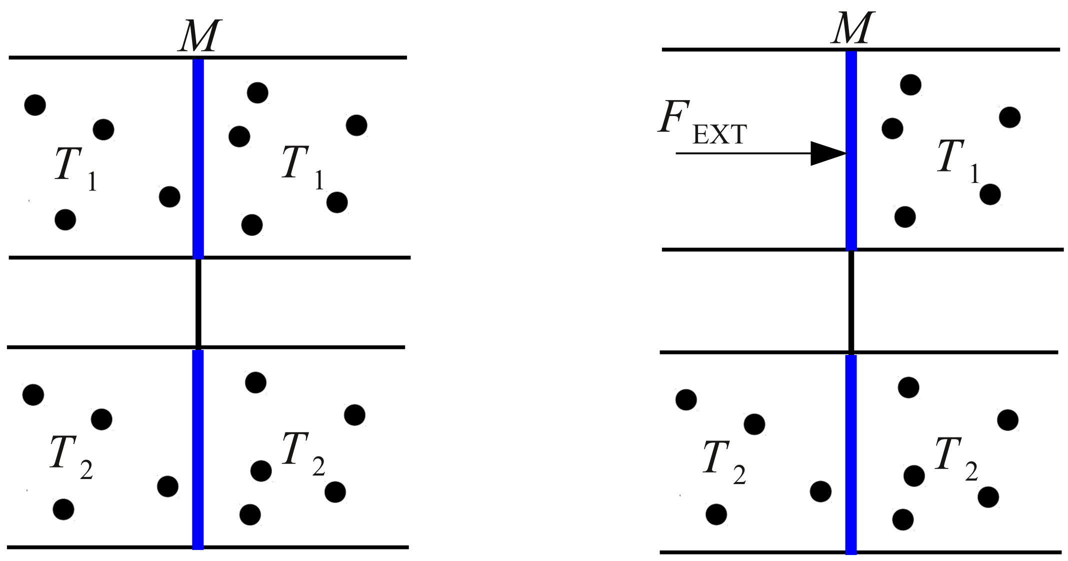




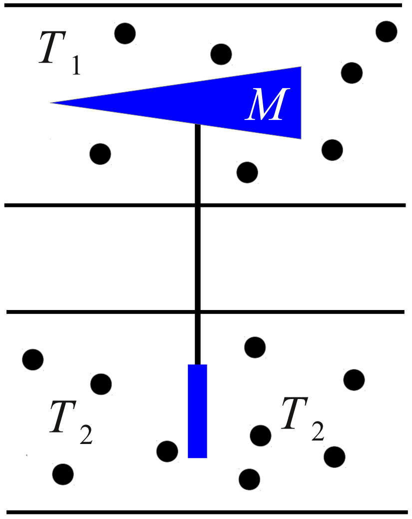


Replace a “ideal” heat bath with a “realistic reservoir”.

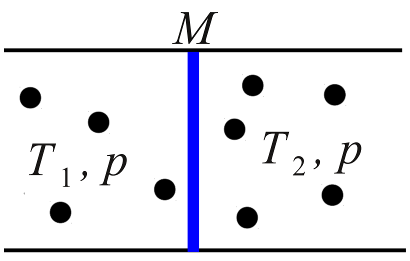
Shared pistons



Ratchet

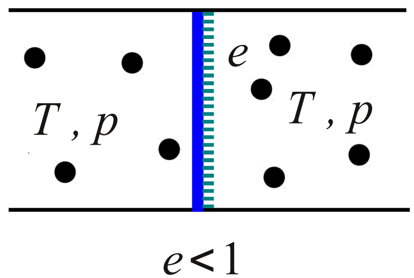


Adiabatic piston



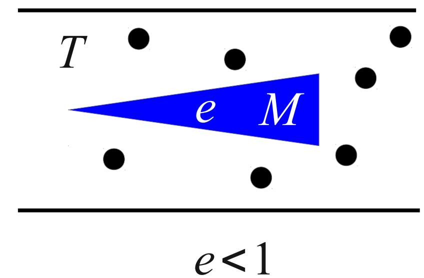
Granular Brownian objects

Inelastic piston



Costantini et al. EPL 82,50008 (2008)

Granular ratchet



Costantini et al. PRE 75, 061124 (2007)
Cleuren and Van den Broeck, EPL 77, 50003 (2007)
Talbot et al. PRE 82, 011135 (2010)

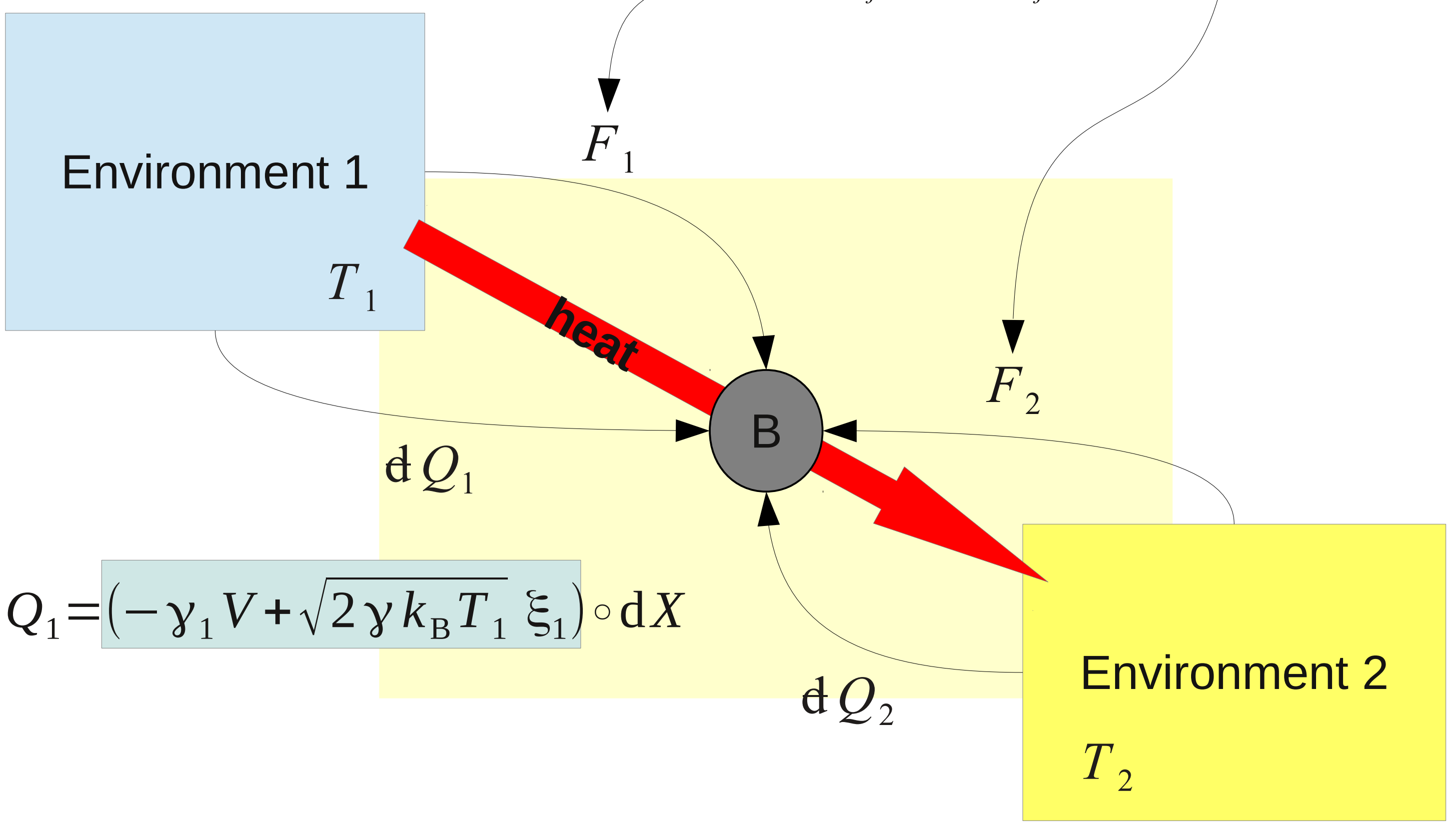
Internal degrees act as the second environment.

Stochastic Thermodynamics: Non-equilibrium Steady State (NESS)

$$M\dot{V} \;=\; -\nabla U(X) \quad -\gamma_1 V+\sqrt{2\gamma_1 k_\mathrm{B} T_1}\;\xi_1 \quad -\gamma_2 V+\sqrt{2\gamma_2 k_\mathrm{B} T_2}\;\xi_2$$

$$\langle \xi_i(t)\rangle = 0$$

$$\langle \xi_i(t)\,\xi_j(t')\rangle = \delta_{ij}\,\delta(t-t')$$

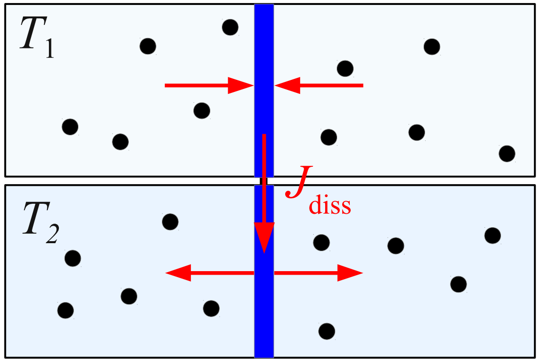


$$\text{đ}Q_1 = \left(-\gamma_1 V+\sqrt{2\gamma k_\mathrm{B} T_1}\;\xi_1\right)\circ \mathrm{d}X$$

$$\text{đ}Q_2 = \left(-\gamma_2 V+\sqrt{2\gamma_2 k_\mathrm{B} T_2}\;\xi_2\right)\circ \mathrm{d}X$$

Sekimoto, *Stochastic Energetics* (Springer, 2010).

Shared Brownian piston



$$M\dot{V} = -\gamma_1 V + \sqrt{2\gamma_1 k_B T_1}\,\xi_1 \quad -\gamma_2 V + \sqrt{2\gamma_2 k_B T_2}\,\xi_2$$

$$M\langle\dot{V}\rangle = -(\gamma_1+\gamma_2)\langle V\rangle, \qquad \langle V\rangle \to 0$$

Heat flow through the fluctuation of the pistons' velocity.

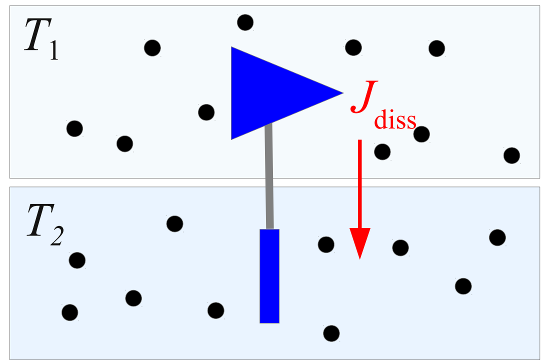
$$\frac{M}{2}\langle V^2\rangle = \frac{k_B T_{\mathrm{kin}}}{2}, \quad \text{where} \quad T_{\mathrm{kin}} = \frac{\gamma_1 T_1 + \gamma_2 T_2}{\gamma_1+\gamma_2}$$

$$J_{\mathrm{diss}} = \frac{kT_1 - kT_2}{M\left(\gamma_1^{-1}+\gamma_2^{-1}\right)}$$

Stochastic Thermodynamics

Sekimoto (1997), Parrondo and Española (1996), Van den Breock *et. al.* (2001)

Brownian Motor



It moves!

$$\langle V \rangle > 0 \quad (T_1 < T_2)$$

$$\langle V \rangle < 0 \quad (T_1 > T_2)$$

$$M\dot{V} = -\gamma_1 V + \sqrt{2\gamma_1 k_B T_1}\,\xi_1 \quad -\gamma_2 V + \sqrt{2\gamma_2 k_B T_2}\,\xi_2$$

$$M\langle \dot{V} \rangle = -(\gamma_1 + \gamma_2)\langle V \rangle, \qquad \langle V \rangle \rightarrow 0$$

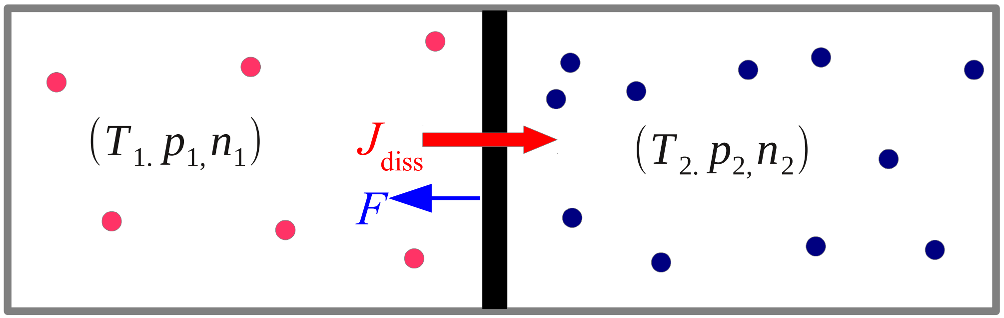
$$M\langle \dot{V} \rangle = -(\gamma_1 + \gamma_2)\langle V \rangle + c\left(\langle V^2 \rangle - \frac{k_B T_1}{M}\right)$$

Boltzmann-master eq. predicts this additional force.

$$\frac{M}{2}\langle V^2 \rangle = \frac{k_B T_{\text{kin}}}{2} \quad \longrightarrow \quad c\left(\frac{k_B T_{\text{kin}}}{M} - \frac{k_B T_1}{M}\right) \propto J_{\text{diss}}$$

Van den Broeck, R.K. and Meurs(2004), PRL **93** 090601

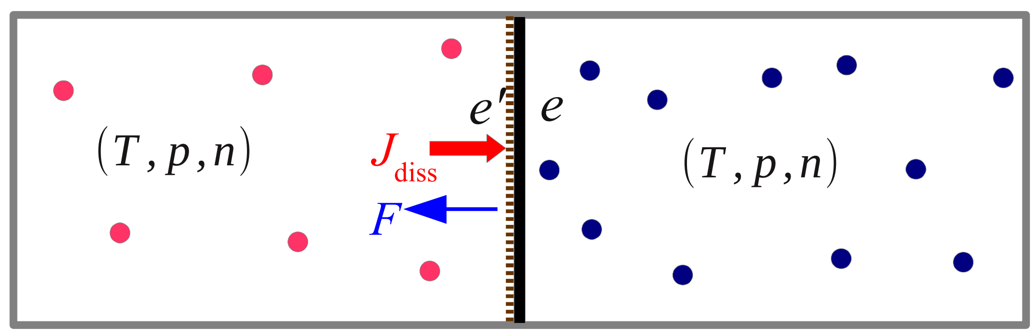
Adiabatic Piston



$$p_1 = p_2\,,\quad T_1 > T_2\,,\quad n_1 < n_2$$

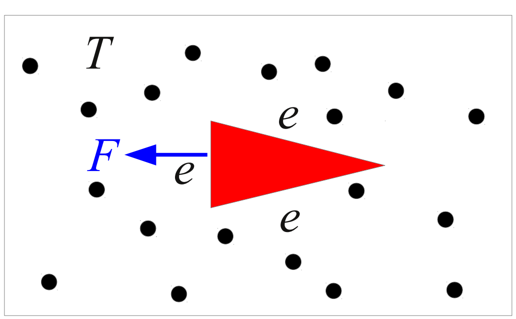
$$p_i = n_i k_B T_i$$

Inelastic Piston

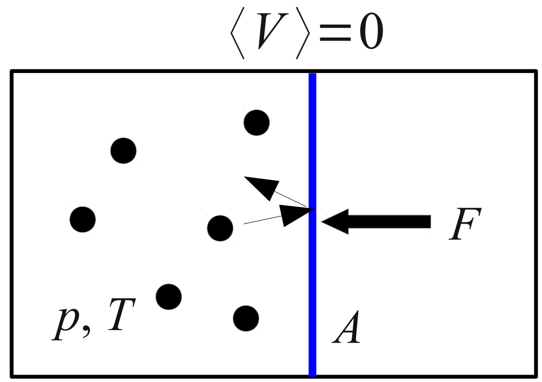


$$e' < e$$

Granular ratchet

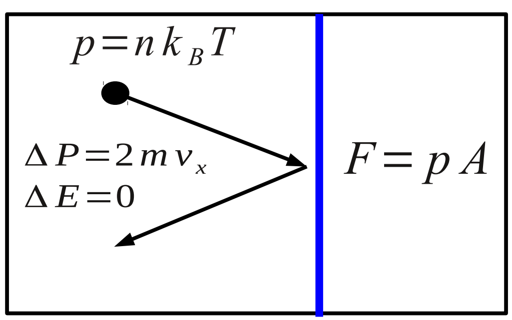


Is force on a wall $F=pA$?

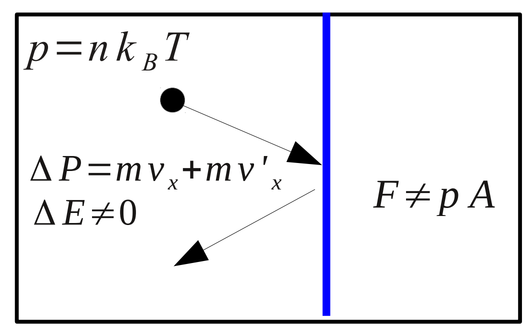


p = hydrostatic pressure

Thermally Equilibrium

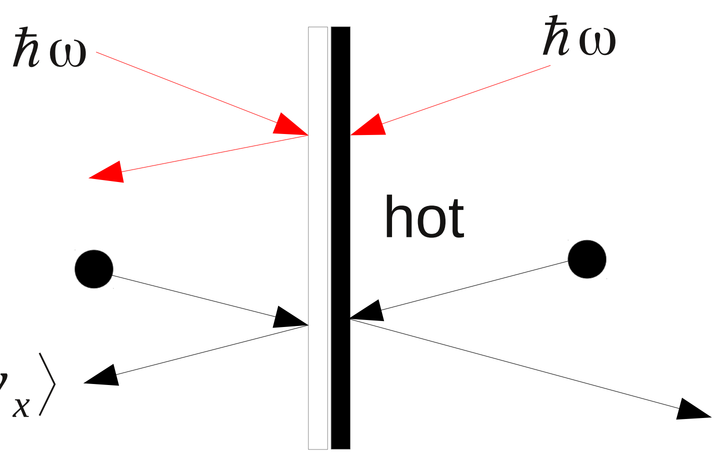


Thermally Non-equilibrium



Macroscopic example: Radiometer

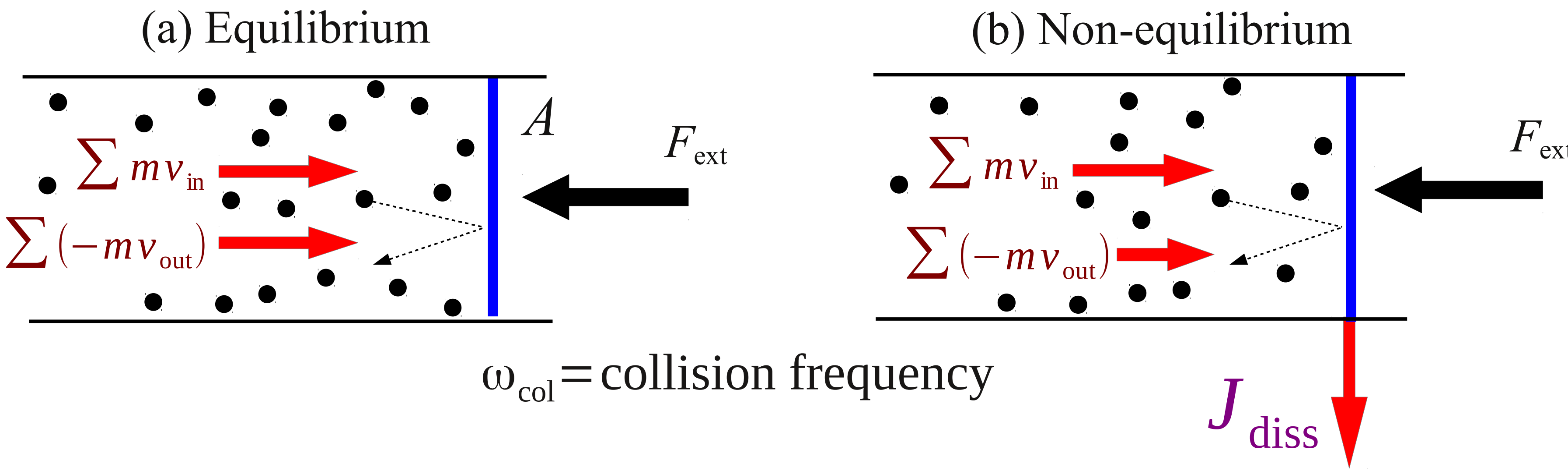
J. Clerk Maxwell
Phil. Trans. R. Soc. Lond. **170** (1879), 231



$$\langle \Delta E \rangle = 0$$
$$\langle \Delta P \rangle = 2m\langle v_x \rangle$$

$$\langle \Delta E \rangle > 0$$
$$\langle \Delta P \rangle = m\langle v_x \rangle + m\langle v'_x \rangle > 2m\langle v_x \rangle$$

Momentum Deficit due to Dissipation (MDD)



$\omega_{col} =$ collision frequency

$$-F = (m v_{th} + m|v'|)\omega_{col} = (2 m v_{th} + m|v'| - m v_{th})\omega_{col} = p A + F_{MDD}$$

$$F_{MDD} = (m|v'| - m v_{th})\omega_{col} \qquad p A = 2 m v_{th} \omega_{col}$$

$$\left(\frac{1}{2} m v_{th}^2 - \frac{1}{2} m|v'|^2\right)\omega_{col} = J_{diss} \xrightarrow[v_{th} \sim |v|]{} (m v_{th} - m|v'|)\omega_{col} \approx \frac{J_{diss}}{v_{th}}$$

This agrees with the result of lengthy calculation of Boltzmann-Master eq. →

$$F_{MDD} \approx -c \frac{J_{diss}}{v_{th}}$$

$c = \sqrt{\frac{\pi}{8}}$ for hard disk gas

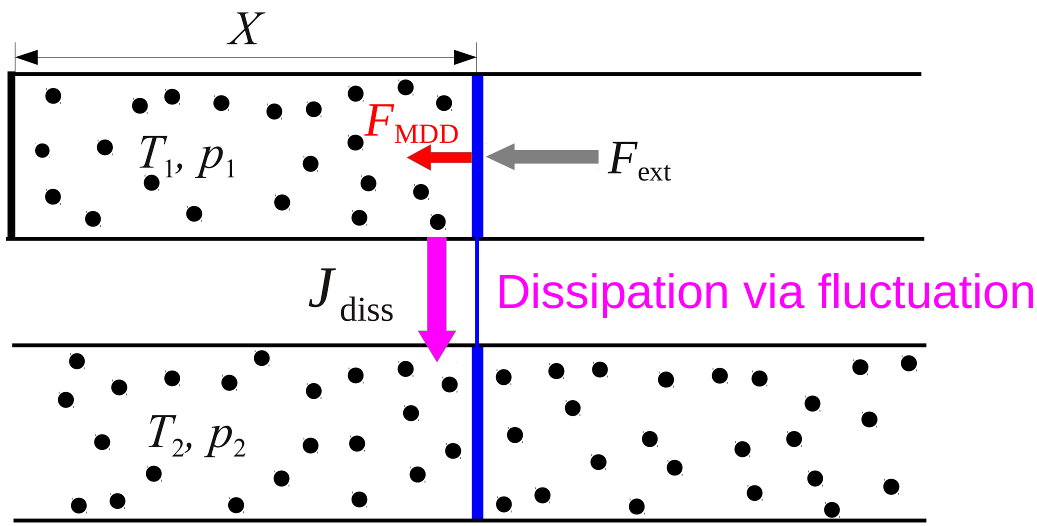
Claim

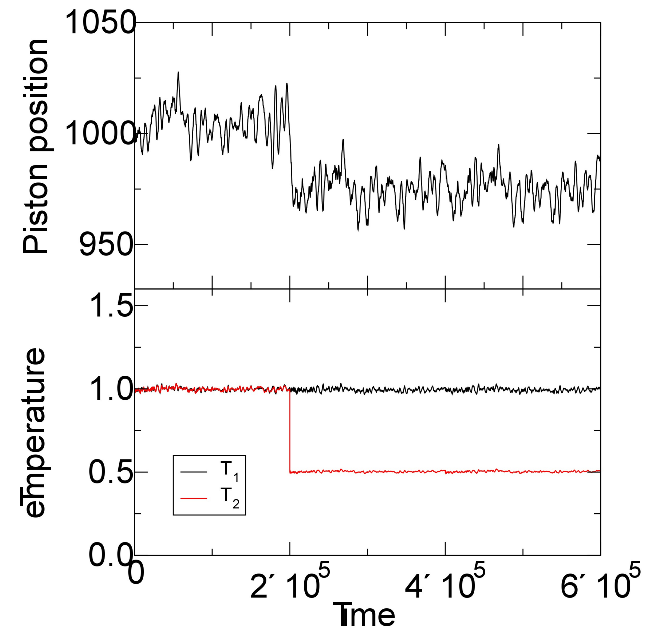
- When heat dissipates through the motion of a Brownian object, the environment exerts a force on the Brownian object.
- Its direction is opposite to the direction of the heat flux.
- Its magnitude is proportional to the heat flux, more specifically

$$F_{\mathrm{MDD}} \approx -c \frac{J_{\mathrm{diss}}}{v_{\mathrm{th}}} \qquad c > 0$$

$$M \langle \dot{V} \rangle = -(\gamma_1 + \gamma_2) \langle V \rangle - c \frac{J_{\mathrm{diss}}}{v_{\mathrm{th}}}$$

Simple Model 1: Shared Brownian Piston

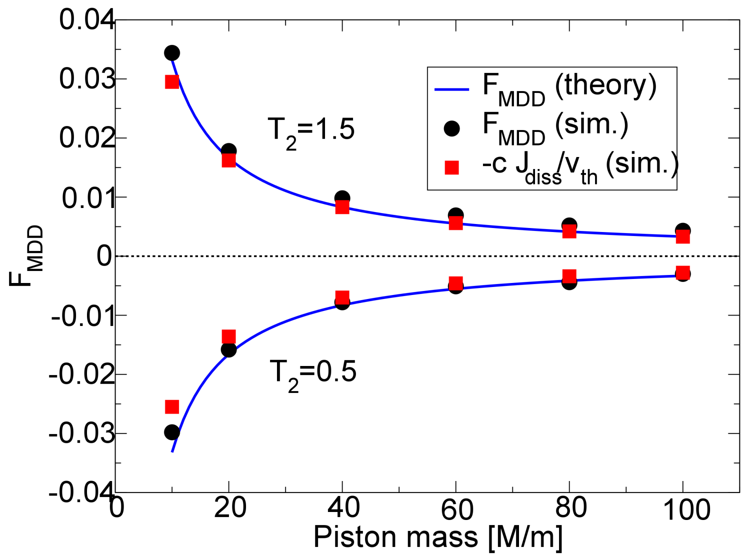




$$F_{\mathrm{MDD}} = -\sqrt{\frac{\pi}{8}} \frac{J_{\mathrm{diss}}}{v_{\mathrm{th}}}$$

$$J_{\mathrm{diss}} = \sqrt{\frac{\pi}{8}} \frac{kT_1 - kT_2}{M\left(\gamma_1^{-1} + \gamma_2^{-1}\right)}$$

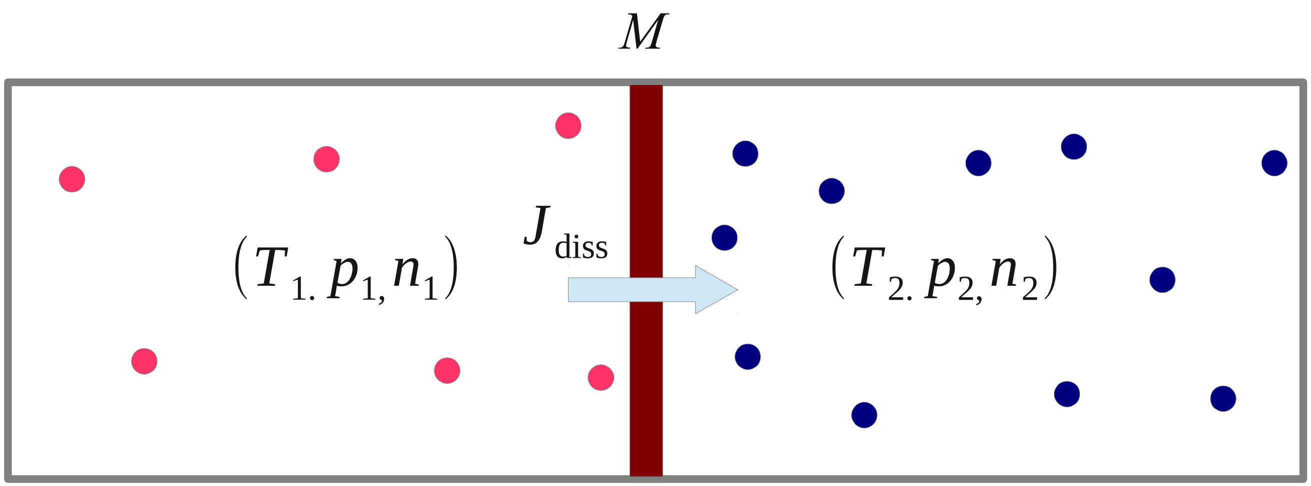
$$\frac{F_{\mathrm{MDD}}}{L} = -\frac{2\rho_1\rho_2}{\rho_1 + 2\rho_2} \frac{m}{M} \left(kT_1 - kT_2\right)$$



Contents

- Introduction: Langevin force and Stochastic Thermodynamics
- Under a certain non-equilibrium condition, the environment exerts a force on a Brownian particle which cannot be explained by the Langeivn theory.
  - Examples:
    - Brownian ratchet, Adiabatic piston,
    - Inelastic Piston, Granular ratchet
- Momentum Deficit due to Dissipation (MDD) is introduced.
- Force caused by MDD explains it all.
- The environment adjusts itself to MDD and it is no longer equilibrium.
- Sasa's Paradox

Adiabatic Piston



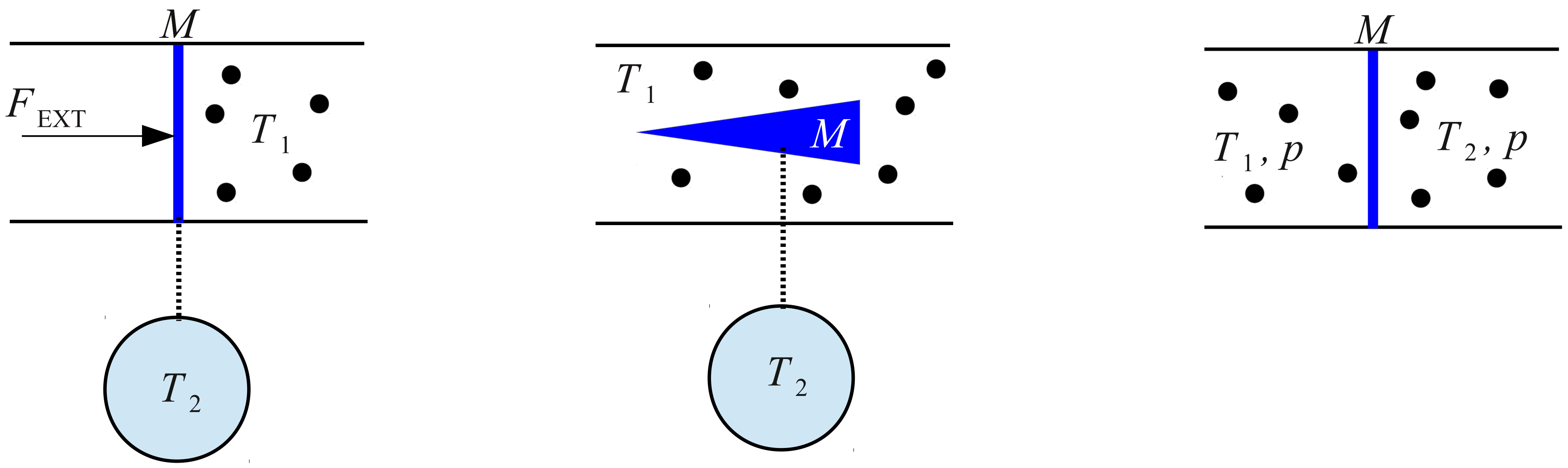
$$p_1 = p_2\,,\quad T_1 > T_2\,,\quad n_1 < n_2$$

$$F_{\text{MDD},1} = -\sqrt{\frac{\pi}{8}}\frac{J_{\text{diss}}}{v_{\text{th},1}} \qquad F_{\text{MDD},2} = \sqrt{\frac{\pi}{8}}\frac{(-J_{\text{diss}})}{v_{\text{th},2}}$$

$$F_{\text{NET}} = -\sqrt{\frac{\pi}{8}} J_{\text{diss}} \left(\frac{1}{v_{\text{th},2}} + \frac{1}{v_{\text{th},1}} \right) = \sqrt{\frac{\pi}{8}} \frac{kT_1 - kT_2}{M(\gamma_1^{-1} + \gamma_2^{-1})} \left(\frac{1}{v_{\text{th},1}} + \frac{1}{v_{\text{th},2}} \right)$$

The piston moves in the opposite direction of the heat.

What kind of reservoir provides F_{MDD} ?

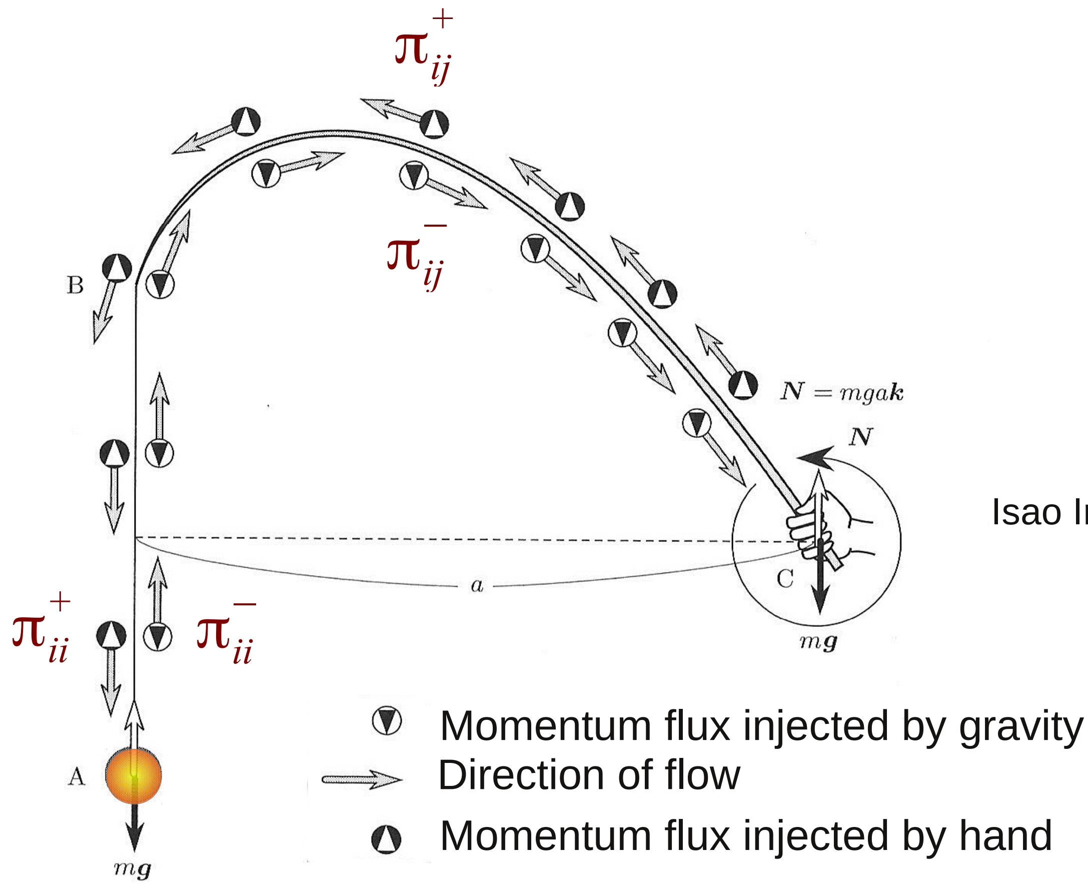


$$M\langle\dot{V}\rangle = -(\gamma_1+\gamma_2)\langle V\rangle - c\,\frac{J_{\mathrm{diss}}}{v_{\mathrm{th}}}$$

$$M\dot{V} = -\gamma_1 V + \sqrt{2\gamma_1 k_B T_1}\,\xi_1 - \gamma_2 V + \sqrt{2\gamma_2 k_B T_2}\,\xi_2 + \text{Non-linear ?}$$

(Momentum Reservoir)

Plyukhin and Schofield, PRE 69, 021112 (2004)



Isao Imai (2003)

$$\pi_{ii} = \pi_{ii}^{+} + \pi_{ii}^{-} = 2\pi_{ii}^{+} = \frac{-mg}{A} \qquad \left(\pi_{ii}^{+} = \pi_{ii}^{-}\right) \quad \text{Equilibrium}$$

$$\pi_{ii} = \pi_{ii}^{+} + \pi_{ii}^{-} = 2\pi_{ii}^{+} - \left(\pi_{ii}^{+} - \pi_{ii}^{-}\right) \qquad \left(\pi_{ii}^{+} \neq \pi_{ii}^{-}\right)$$

Momentum Deficit

Momentum Deficit due to Dissipation in Hydrodynamics

$$\rho = \int m f(\boldsymbol{x}, \boldsymbol{c}) \mathrm{d}\boldsymbol{c}$$ mass density

$$\rho v_i = \int m c_i f(\boldsymbol{x}, \boldsymbol{c}) \mathrm{d}\boldsymbol{c} = 0$$ momentum density

$$p = \frac{1}{3} \mathrm{Tr}\, \pi = n k_\mathrm{B} T$$

$$\rho u = \int \frac{m}{2} \boldsymbol{c} \cdot \boldsymbol{c} f(\boldsymbol{x}, \boldsymbol{c}) \mathrm{d}\boldsymbol{c} = \frac{3}{2} n k_\mathrm{B} T$$ energy density

$$\tilde{\pi}_{ij} = \pi_{ij} - p \delta_{ij}$$

$$\pi_{ij} = \int m c_i c_j f(\boldsymbol{x}, \boldsymbol{c}) \mathrm{d}\boldsymbol{c}$$ pressure tensor (momentum flux)

$$q_i = \int \frac{m}{2} \boldsymbol{c} \cdot \boldsymbol{c} c_i f(\boldsymbol{x}, \boldsymbol{x}) \mathrm{d}\boldsymbol{c}$$ heat flux

$$f(\boldsymbol{x}, \boldsymbol{c}) = n \left(\frac{\beta}{\pi}\right)^{3/2} \mathrm{e}^{-\beta \boldsymbol{c} \cdot \boldsymbol{c}} \left[1 + \frac{\beta}{p} \left\{ \boldsymbol{c} \cdot \left(\tilde{\pi} + 4 \frac{\beta}{5} \boldsymbol{q} \otimes \boldsymbol{c} \right) \cdot \boldsymbol{c} - 2 \boldsymbol{q} \cdot \boldsymbol{c} \right\} \right] \qquad \beta \equiv \frac{m}{2 k_\mathrm{B} T}$$

$$\pi_{ii}^{+} = \int_{c_i > 0} m c_i c_i f(\boldsymbol{x}, \boldsymbol{c}) \mathrm{d}\boldsymbol{c} = \frac{\pi_{ii}}{2} + \frac{1}{5} \sqrt{\frac{2}{\pi}} \frac{q_i}{v_\mathrm{th}}$$

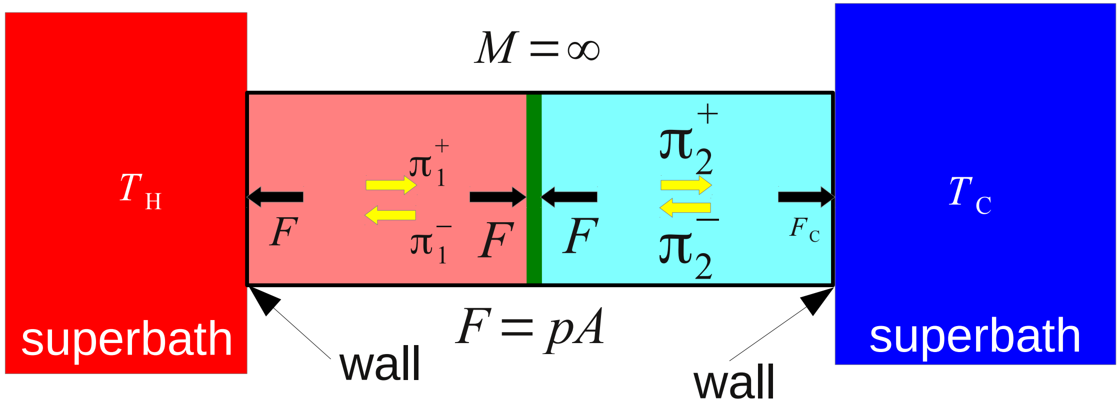
$$\pi_{ii}^{-} = \int_{c_i < 0} m c_i c_i f(\boldsymbol{x}, \boldsymbol{c}) \mathrm{d}\boldsymbol{c} = \frac{\pi_{ii}}{2} - \frac{1}{5} \sqrt{\frac{2}{\pi}} \frac{q_i}{v_\mathrm{th}}$$

$$\pi_{ii}^{-} - \pi_{ii}^{+} = -\frac{1}{5} \sqrt{\frac{8}{\pi}} \frac{q_i}{v_\mathrm{th}}$$

$$F_\mathrm{MDD} \approx -c \frac{J_\mathrm{diss}}{v_\mathrm{th}}$$

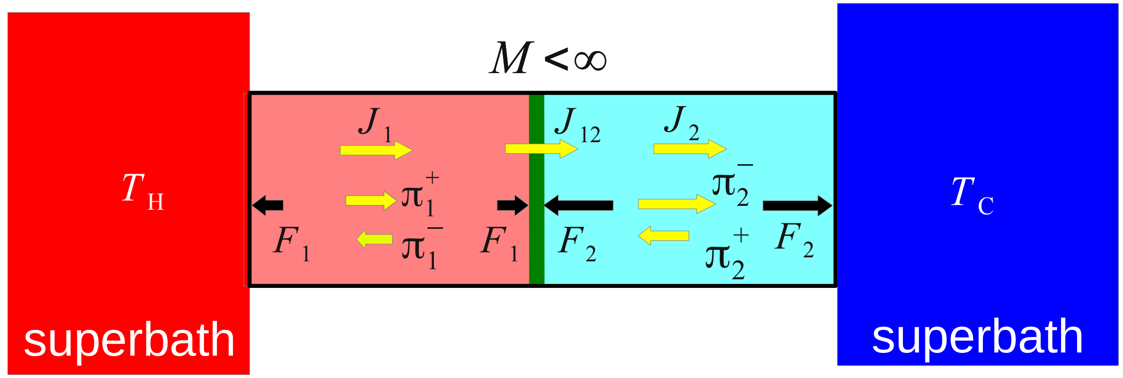
What is happening in the reservoir?

No dissipation



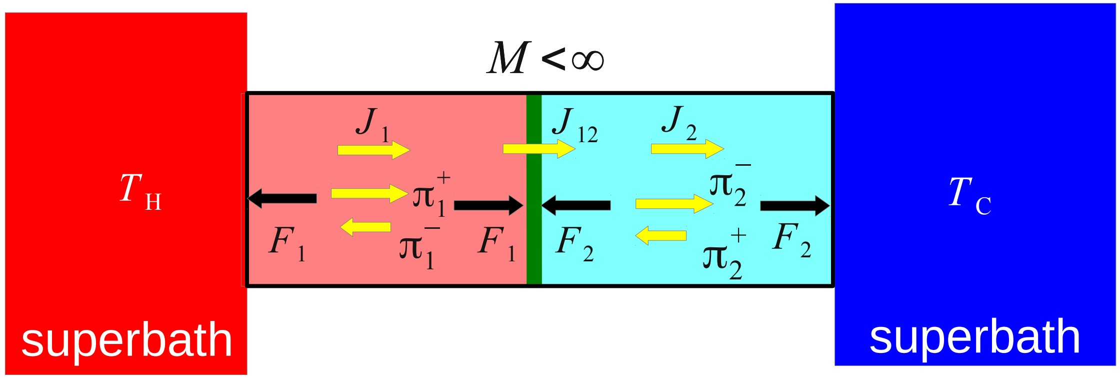
$$p = \pi_1^+ + \pi_1^- = 2\pi_1^+$$
$$= \pi_2^+ + \pi_2^- = 2\pi_2^-$$

“Ideal Gas” model



$$p = 2\pi_1^+ = 2\pi_2^-$$

More Realistic Interpretation

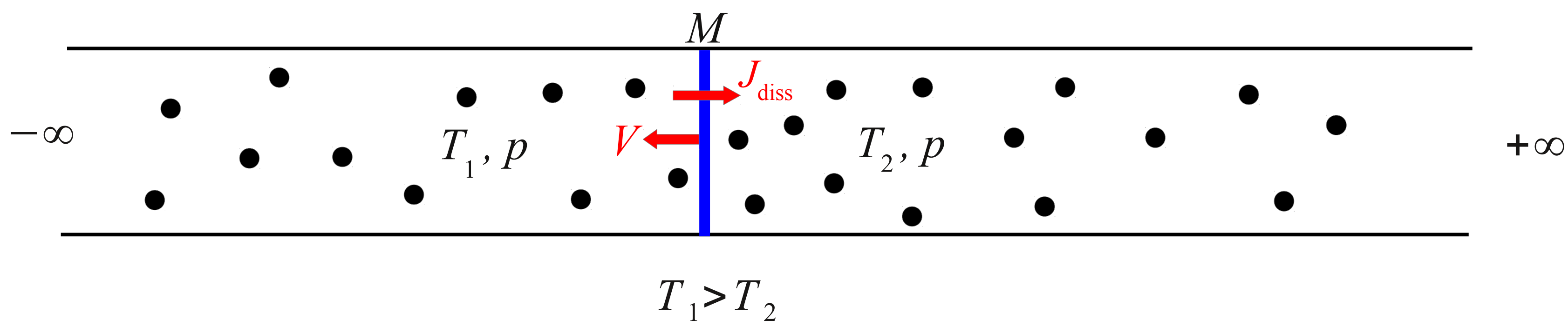


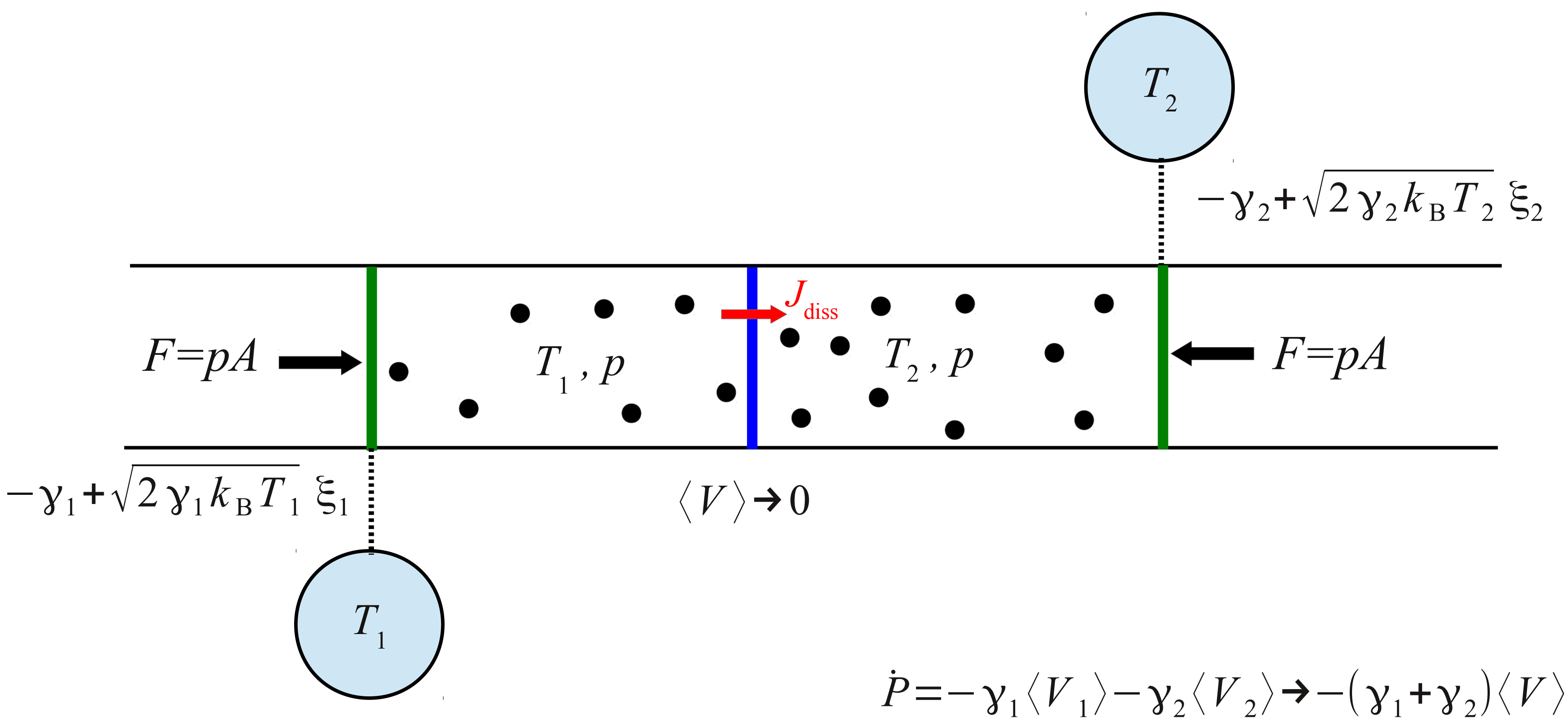
$$\pi_1^+ + \pi_1^- = \pi_2^+ + \pi_2^-$$

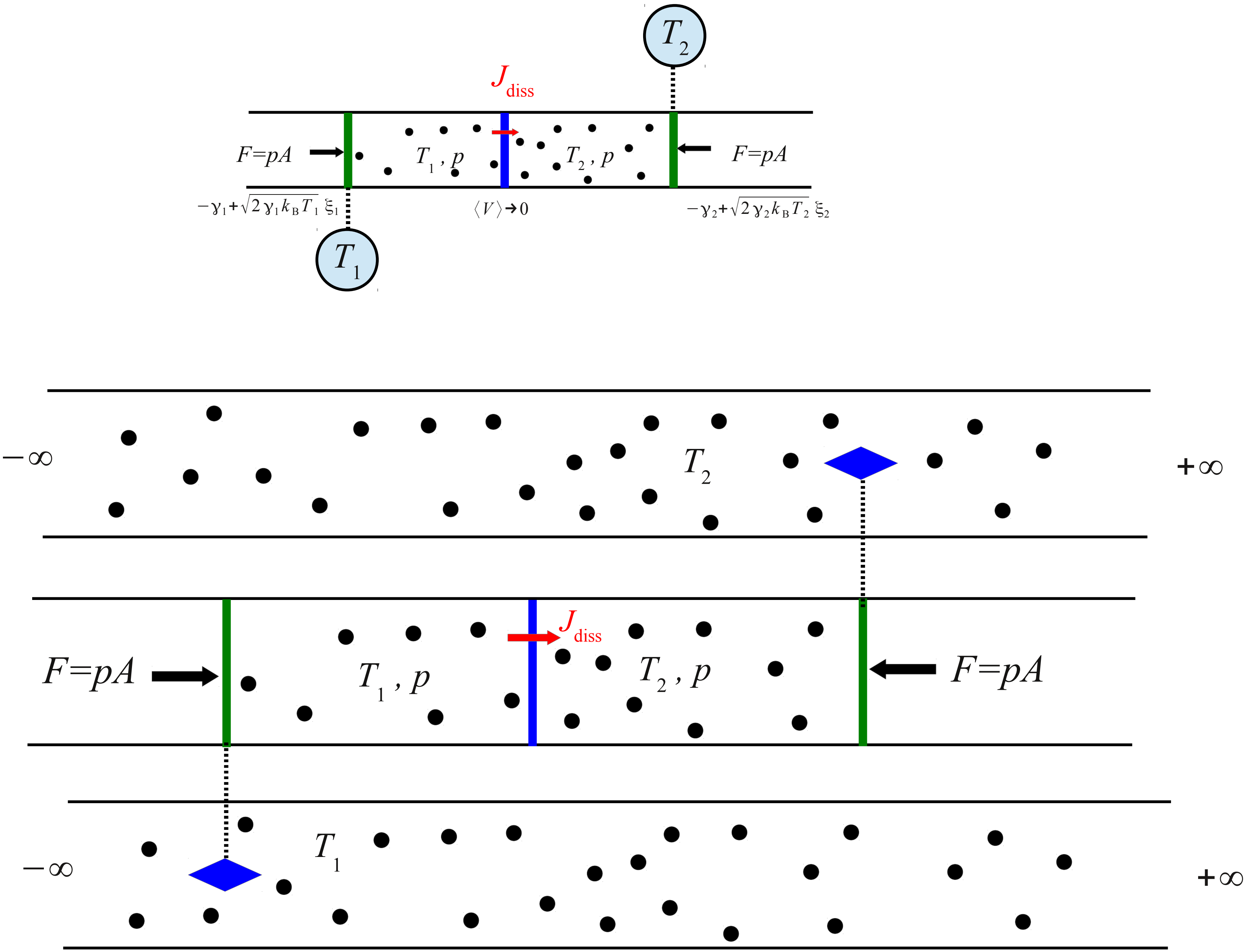
$$F_{MDD} - (\gamma_1 + \gamma_2) V = 0$$

Sasa's paradox of adiabatic piston

Steady state motion

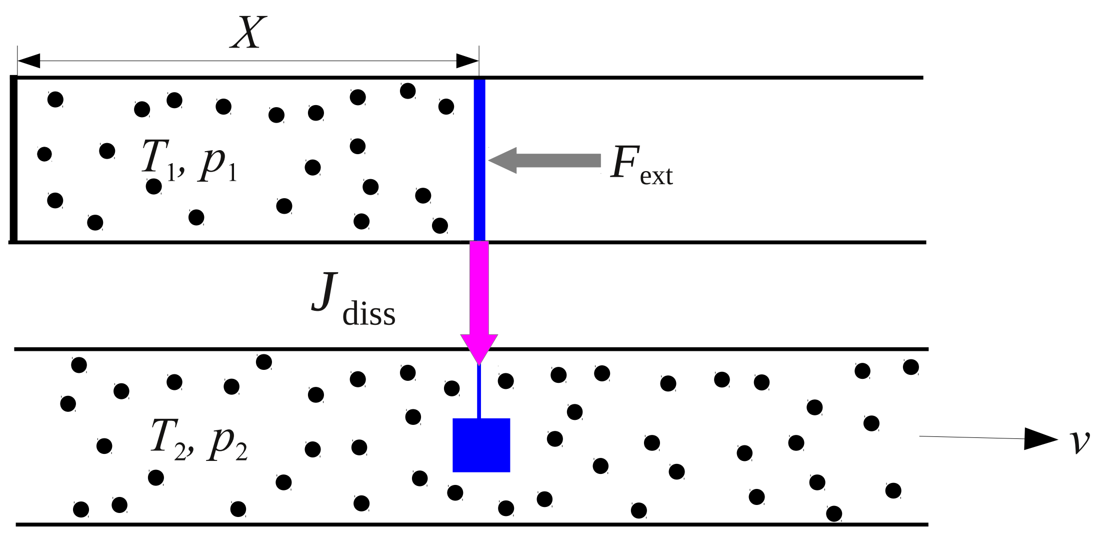


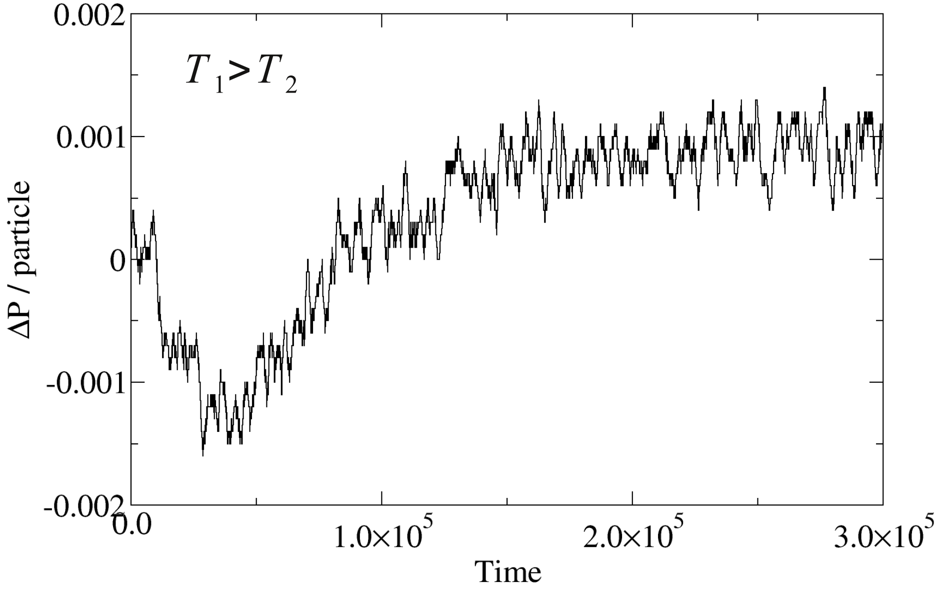




T2
Jdiss
F=pA
T1 , p
T2 , p
F=pA
−γ1+√(2γ1kBT1) ξ1
⟨V⟩→0
−γ2+√(2γ2kBT2) ξ2
T1
−∞
T2
+∞
Jdiss
F=pA
T1 , p
T2 , p
F=pA
−∞
T1
+∞

Momentum Bath





Conclusions

- Concept of Momentum Deficit due to Dissipation (MDD)
  Is introduced.

- Force by MDD $$F_{\mathrm{MDD}} = -c \frac{J_{\mathrm{diss}}}{v_{\mathrm{th}}}$$

- MDD captures the asymmetry in fluctuation.

- Adiabatic piston, Brownian ratchets, Inelastic piston, and
  Granular ratchet ... can be all intuitively and quantitatively explained
  by MDD without lengthy calculation.

- In non-equilibrium, the environment can provide momentum along with heat
  if the system request it.

- We don't know a simple stochastic model for such an environment.
  Bolzmann-master eq., multi-Poisson processes, non-linear Langevin, ….