Efficiency at Maximum Power in Weak Dissipation Regimes

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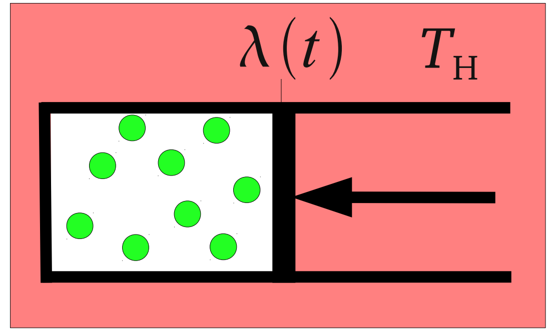
- Introduction: Efficiency at maximum power
- General derivation of the efficiency at maximum power with a weak dissipation approximation
- A case study: Quantum dot heat engine

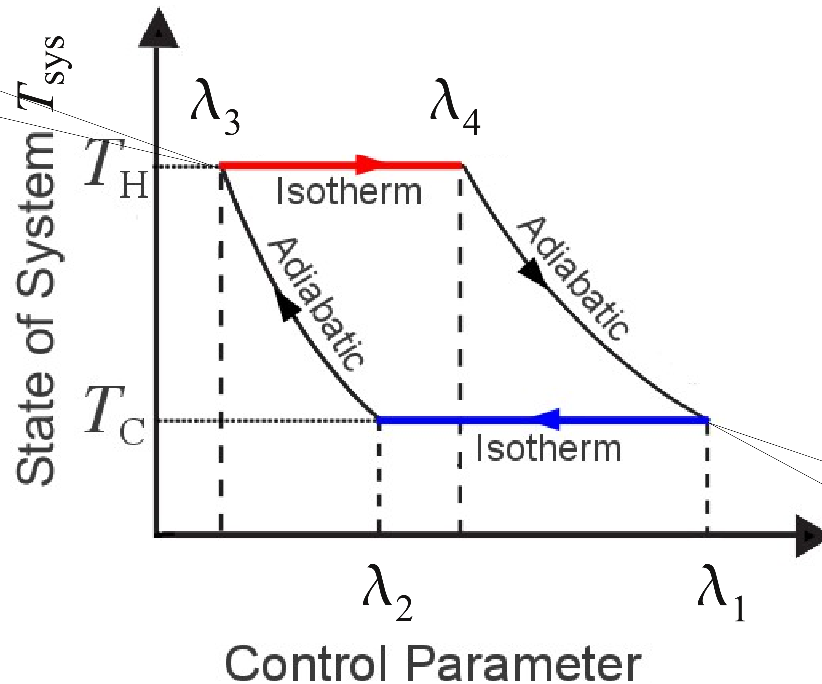
Esposito, Kawai, Van den Broeck, Lindenberg, PRL **105**, 150603 (2010)
Esposito, Kawai, Van den Broeck, Lindenberg, PRE **81**, 041106 (2010)

Carnot Cycle



Sadi Carnot

Stop the piston when $T_{sys}=T_H$.



Stop the piston when $T_{sys}=T_C$.

1st Law: $W_{Net}+Q_H+Q_C=0$

2nd law: $\eta=\dfrac{-W_{Net}}{Q_H} \;=\; \leq\; 1-\dfrac{T_C}{T_H}\equiv\eta_C$

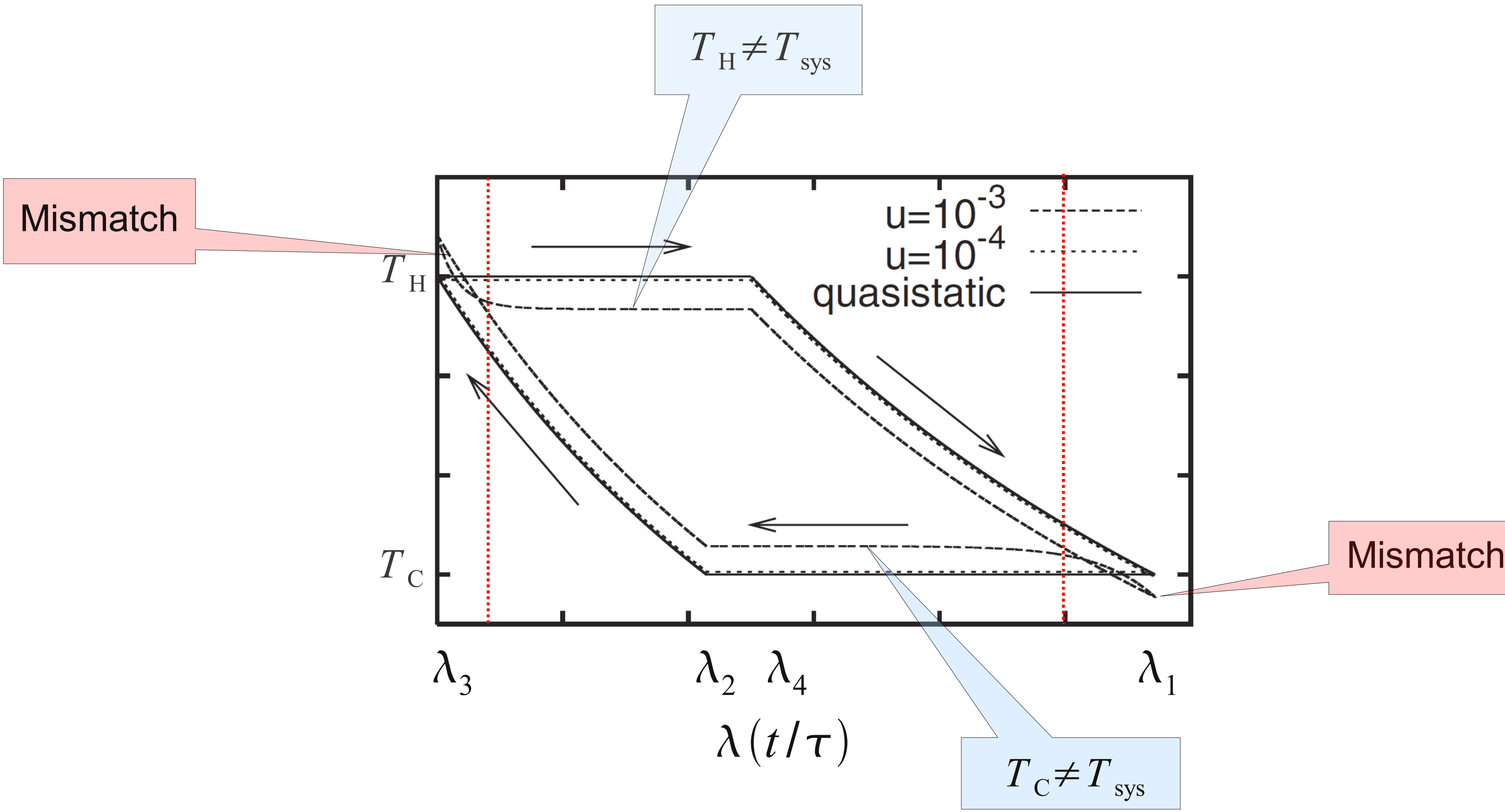
Quasi static (reversible) limit $\tau\rightarrow\infty$

$\eta\rightarrow\eta_C$ (Highest efficiency)

$P=\dfrac{-W}{\tau}\rightarrow 0$ (Lowest power)

Finite Time Thermodynamics

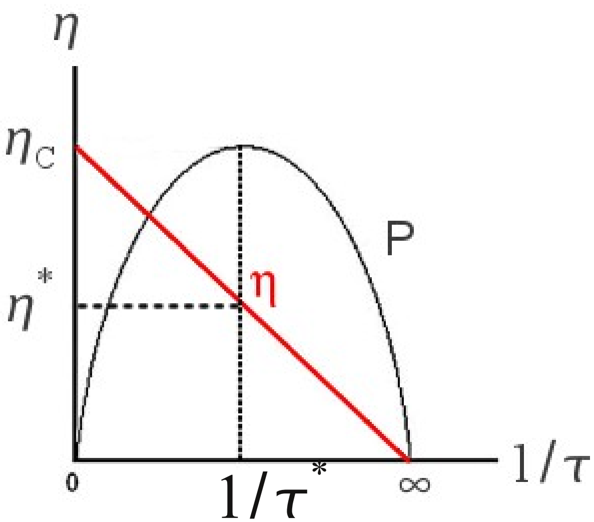
What will happen if τ is finite? (beyond $\eta < \eta_C$)



Izumida and Okuda, EPL 83 (2008), 60003

Efficiency at maximum power

Efficiency decreases as τ increases.
But Power reaches its maximum at a certain τ.



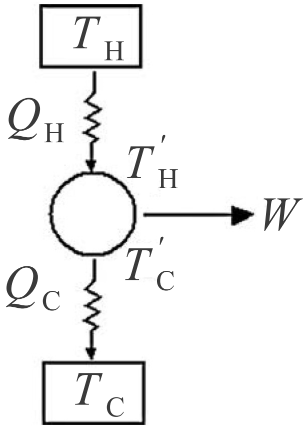
maximum power: $$P^* = \frac{-W}{\tau} = \frac{Q_H(\tau_H^*) + Q_C(\tau_C^*)}{\tau_H^* + \tau_C^*}$$

efficiency at maximum power: $$\eta^* = 1 + \frac{Q_C(\tau_C^*)}{Q_H(\tau_H^*)}$$

Maximize P with respect to ?

- only τ assuming $\lambda(t/\tau)$
- the form of $\lambda(t)$
- $\lambda(t)$ and other system parameters

Curzon-Ahlborn efficiency



Efficiency at maximum power

$$\eta_{CA} = 1 - \sqrt{\frac{T_C}{T_H}}$$

Novikov, J. Nucl. Energy **II** (1958), 125

Curzon-Ahlborn, Am. J. Phys **43** (1975), 22

TABLE I. Observed performance of real heat engines.

Power source	T_C	T_H	η_C	η_{CA}	$\eta_{Observed}$
West Thurrock (U.K.)[2] Coal Fired Steam Plant	~25	565	64.1%	40%	36%
CANDU (Canada)[4] PHW Nuclear Reactor	~25	300	48.0	28%	30%
Larderello (Italy)[5] Geothermal Steam Plant	80	250	32.3%	17.5%	16%
Steam power plant (USA)	298	923	67.6%	43.2%	40%

How universal is the Curzon-Ahlbone efficiency?

Many case studies:

$$\eta^* \approx \eta_{CA} = 1 - \sqrt{\frac{T_C}{T_H}} = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{\eta_C^3}{16} + \cdots$$

Brownian heat engine [Schmiedl&Seifert, EPL **81** (2008), 20003]

$$\eta^* = \frac{2(T_H - T_C)}{3T_H + T_C} = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{\eta_C^3}{32} + \ldots$$

Linear non-equilibrium thermodynamics+strong coupling

$$\eta^* = \frac{\eta_C}{2}$$

Van den Broeck, PRL **95** (2005), 190602

Non-linear correction (with left-right symmetry)+strong coupling

$$\eta^* = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + o(\eta_C^3)$$

Esposito *et al*., PRL **102** (2009), 130602

Weak dissipation limit

Change in system entropy

$$\Delta S = \Delta S^{\mathrm{e}} + \Delta S^{\mathrm{i}}$$

Irreversible entropy production

Entropy flow: $\Delta S^{\mathrm{e}} = \dfrac{Q}{T}$

Quasi static limit $\tau \to \infty : \ \Delta S^{\mathrm{e}} \to \Delta S^{\mathrm{rev}}, \quad \Delta S^{\mathrm{i}} \to 0$

Asymptotic expansion (Weak dissipation approximation)

$$\Delta S^{\mathrm{e}} = \Delta S^{\mathrm{rev}} - \frac{\Sigma^{\mathrm{e}}}{\tau} + o\left(\frac{1}{\tau^2}\right)$$

$$\Delta S^{\mathrm{i}} = -\frac{\Sigma^{\mathrm{i}}}{\tau} + o\left(\frac{1}{\tau^2}\right)$$

General case:

$$\frac{d}{dt}|P(t)\rangle = \hat{W}^T(t)|P(t)\rangle$$

$$\sum_m W^T_{mn}(t) = 0$$

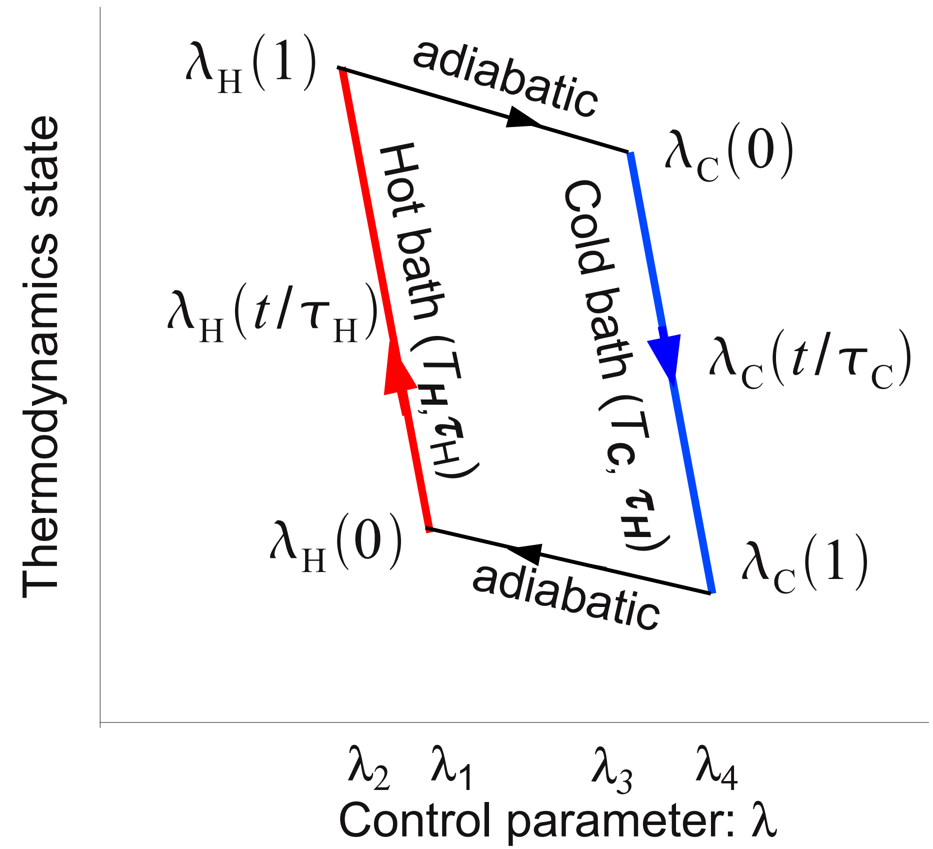
$$\hat{W}^T(t)|P^{qs}(t)\rangle = 0,$$

$$W^T_{mn}(t)P^{qs}_n(t) = W^T_{nm}(t)P^{qs}_m(t)$$

Scaling time $s = \frac{t}{\tau}$

$$\hat{W}^T(t) = \hat{W}^T[\lambda(t/\tau)]$$

$$\frac{\mathrm{d}}{\mathrm{d}s}|P_\tau(s)\rangle = \tau\hat{W}^T(s)|P_\tau(s)\rangle$$



Four corners are fixed such that the cycle is a Carnot cycle at the quasi static limit.

Asymptotic expansion

$$|P_\tau(s)\rangle = |P^{\mathrm{qs}}(s)\rangle + \frac{1}{\tau}|\delta^{(1)}(s)\rangle + \frac{1}{\tau^2}|\delta^{(2)}(s)\rangle + o(\tau^{-3})$$

$$\hat{W}^T(s)|\delta^{(1)}(s)\rangle = \frac{\mathrm{d}}{\mathrm{d}s}|P^{\mathrm{qs}}(s)\rangle, \quad \hat{W}^T(s)|\delta^{(2)}(s)\rangle = \frac{\mathrm{d}}{\mathrm{d}s}|\delta^{(1)}(s)\rangle, \quad \sum_m \delta_m^{(1)} = \sum_m \delta_m^{(2)} = 0$$

Entropy flow

$$\Delta S^{\mathrm{e}} = \int_0^\tau \mathrm{d}t \sum_m \sum_n W_{mn}^T(t) P_n(t) \ln\left[\frac{W_{nm}^T(t)}{W_{mn}^T(t)}\right]$$

$$= \Delta S^{\mathrm{rev}} - \frac{1}{\tau}\int_0^1 \mathrm{d}s \sum_m \left[\frac{\mathrm{d}}{\mathrm{d}s}\delta_m^{(1)}(s)\right] \ln P_m^{\mathrm{qs}}(s)$$

$$\Delta S^{\mathrm{rev}} = -k\sum_m P_m^{\mathrm{qs}}(1)\ln P_m^{\mathrm{qs}}(1) + k\sum_m P_m^{\mathrm{qs}}(0)\ln P_m^{\mathrm{qs}}(0)$$

Entropy production

$$\Delta S^{\mathrm{i}} = \int_0^\tau \mathrm{d}t \sum_m \sum_n W_{mn}^T(t) P_n(t) \ln\left[\frac{W_{mn}^T(t) P_n(t)}{W_{nm}^T(t) P_m(t)}\right]$$

$$= -\frac{1}{\tau}\int_0^1 \mathrm{d}s \sum_m \delta_m^{(1)}(s) \frac{\mathrm{d}}{\mathrm{d}s} \ln P_m^{\mathrm{qs}}(s)$$

Asymptotic expansion (Weak dissipation approximation)

$$\Delta S^{\mathrm{e}} = \Delta S^{\mathrm{rev}} - \frac{\Sigma^{\mathrm{e}}}{\tau} + o\left(\frac{1}{\tau^2}\right)$$

Heat

$$Q_{\mathrm{H}} = T_{\mathrm{H}} \Delta S_{\mathrm{H}}^{\mathrm{e}} \approx T_{\mathrm{H}} \Delta S^{\mathrm{rev}} - \frac{T_{\mathrm{H}} \Sigma_{\mathrm{H}}^{\mathrm{e}}}{\tau_{\mathrm{H}}}$$

$$Q_{\mathrm{C}} = T_{\mathrm{C}} \Delta S_{\mathrm{C}}^{\mathrm{e}} \approx -T_{\mathrm{C}} \Delta S^{\mathrm{rev}} - \frac{T_{\mathrm{C}} \Sigma_{\mathrm{C}}^{\mathrm{e}}}{\tau_{\mathrm{C}}}$$

Power

$$P = \frac{Q_{\mathrm{H}} + Q_{\mathrm{C}}}{\tau_{\mathrm{H}} + \tau_{\mathrm{C}}} = \frac{(T_{\mathrm{H}} - T_{\mathrm{C}}) \Delta S^{\mathrm{rev}} - \frac{T_{\mathrm{H}} \Sigma_{\mathrm{H}}^{\mathrm{e}}}{\tau_{\mathrm{H}}} - \frac{T_{\mathrm{C}} \Sigma_{\mathrm{C}}^{\mathrm{e}}}{\tau_{\mathrm{C}}}}{\tau_{\mathrm{H}} + \tau_{\mathrm{C}}}$$

Maximizing power with respect to τ_{H} and τ_{C}

$$\tau_H^* = \frac{2 T_H \Sigma_H^{\mathrm{e}}}{(T_H - T_C) \Delta S^{rev}} \left(1 + \sqrt{\frac{T_C \Sigma_C^{\mathrm{e}}}{T_H \Sigma_H^{\mathrm{e}}}}\right) \qquad \tau_C^* = \frac{2 T_C \Sigma_C^{\mathrm{e}}}{(T_H - T_C) \Delta S^{rev}} \left(1 + \sqrt{\frac{T_H \Sigma_H^{\mathrm{e}}}{T_C \Sigma_C^{\mathrm{e}}}}\right)$$

Efficiency at Maximum Power

$$\eta^* = 1 + \frac{Q_C^*}{Q_H^*} = \frac{\eta_C}{2} + \frac{\eta_C^2}{4\left(1 + \sqrt{\frac{\Sigma_C^{\mathrm{e}}}{\Sigma_H^{\mathrm{e}}}}\right)} + \frac{\eta_C^3}{8\left(1 + \sqrt{\frac{\Sigma_C^{\mathrm{e}}}{\Sigma_H^{\mathrm{e}}}}\right)} + \cdots$$

$$\frac{\Sigma_C^{\mathrm{e}}}{\Sigma_{\mathrm{H}}^{\mathrm{e}}} = \infty \rightarrow \eta^* = \frac{\eta_{\mathrm{C}}}{2}$$

$$\frac{\Sigma_C^{\mathrm{e}}}{\Sigma_{\mathrm{H}}^{\mathrm{e}}} = 0 \rightarrow \eta^* = \frac{\eta_{\mathrm{C}}}{2} + \frac{\eta_{\mathrm{C}}^2}{4} + \frac{\eta_{\mathrm{C}}^3}{8} + \cdots = \frac{\eta_{\mathrm{C}}}{2 - \eta_{\mathrm{C}}}$$

$$\frac{\Sigma_C^{\mathrm{e}}}{\Sigma_{\mathrm{H}}^{\mathrm{e}}} = 1 \rightarrow \eta^* = \frac{\eta_{\mathrm{C}}}{2} + \frac{\eta_{\mathrm{C}}^2}{8} + \frac{\eta_{\mathrm{C}}^3}{16} + \cdots = \eta_{\mathrm{CA}}$$

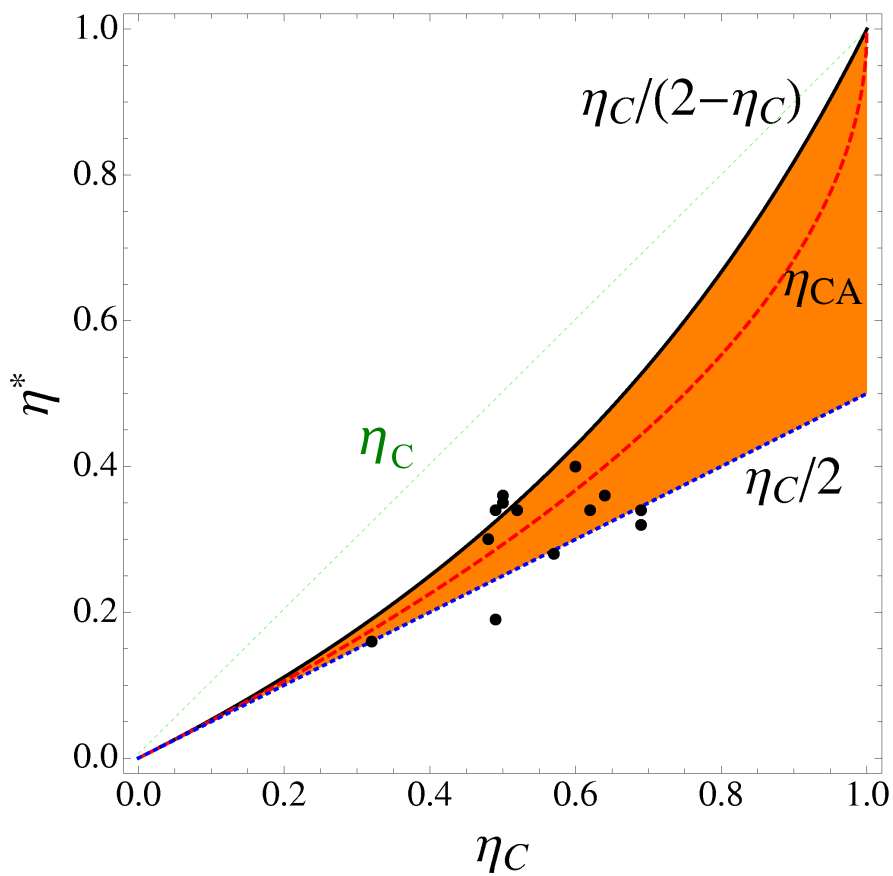
$$\frac{\eta_{\mathrm{C}}}{2} \le \eta^* \le \frac{\eta_{\mathrm{C}}}{2 - \eta_{\mathrm{C}}}$$

↑ η_{CA}

In the middle
of two bounds

TABLE I. Observed performance of real heat engines.

Power source	T_C	T_H	η_C	η_{CA}	$\eta_{Observed}$	Lower bound η_L	Upper bound η_U
West Thurrock (U.K.)[2] Coal Fired Steam Plant	~25	565	64.1%	40%	36%	32%	47%
CANDU (Canada)[4] PHW Nuclear Reactor	~25	300	48.0	28%	30%	24%	31%
Larderello (Italy)[5] Geothermal Steam Plant	80	250	32.3%	17.5%	16%	16%	19%
Steam power plant (USA)	25	650	67.6%	43.2%	40%	34%	51%



I think I have already said so.

T. Schmiedl and U. Seifert, Europhys. Lett. 81, 20003 (2008).

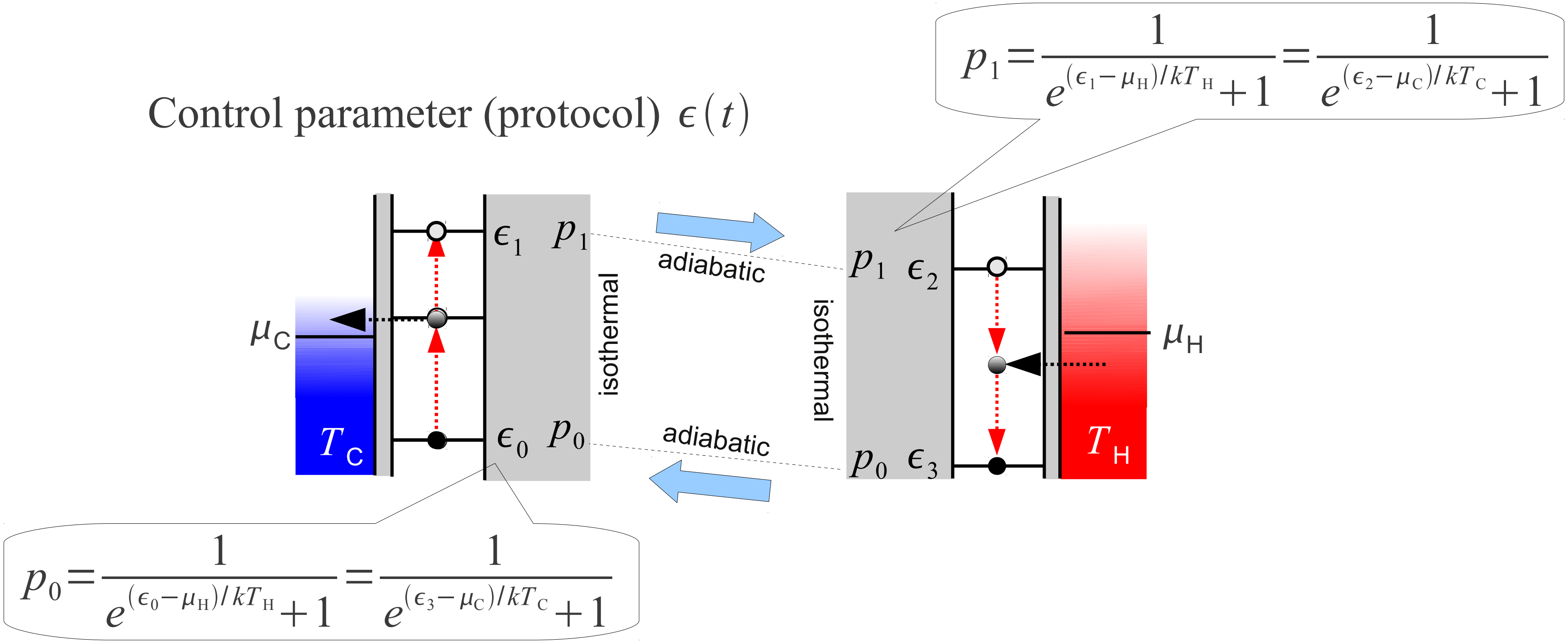
B. Gaveau, M. Moreau, and L. S. Schulman, Phys. Rev. Lett. 105, 060601 (2010).

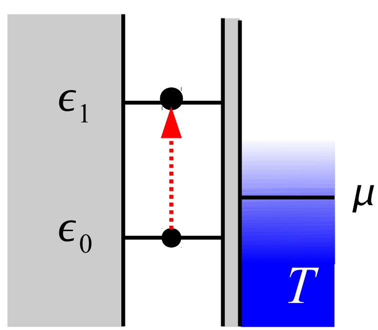
L. Chen and Z. Yan, J. Chem. Phys. 90, 3740 (1989)

S. Velasco *et al*., J. Phys. D 34, 1000 (2001).

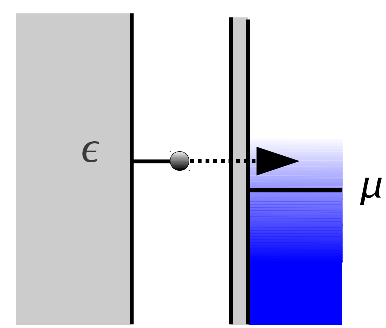
Quantum Dot Carnot Engine

Control parameter (protocol) $\epsilon(t)$





Work done to the system
$W = \epsilon_1 - \epsilon_0$



Heat flows out
$Q = \epsilon - \mu$

Steps to find the efficiency at maximum power

Step 1: Optimize the protocol $(\epsilon(t)$ or $p(t))$

Step 2: Maximize power with respect to τ_H and τ_C

Step 3: Evaluate efficiency at maximum power

Master equation $\dot{p}(t) = -\omega_1 p(t) + \omega_2 [1 - p(t)] \Longrightarrow \dot{p}(t) = -C\, p(t) + \frac{C}{e^{\epsilon(t)/kT} + 1}$

$$\omega_1 = \frac{C}{e^{-\epsilon(t)/kT} + 1}, \quad \omega_2 = \frac{C}{e^{+\epsilon(t)/kT} + 1}$$

C=tunneling rate

Work: $W[p(t)] = \int_0^\tau \dot{\epsilon}(t)\, p(t)\, dt$

Heat: $Q[p(t)] = \int_0^\tau \epsilon(t)\, \dot{p}(t)\, dt$

Optimizing protocol

$$\mathcal{Q}[P(\cdot)] = \int_0^\tau \varepsilon(t)\dot{p}(t)\mathrm{d}t = \int_0^\tau \ln\left[\frac{1}{Cp(t)+\dot{p}(t)} - 1\right]\dot{p}(t)\mathrm{d}t \equiv \int_0^\tau L(p,\dot{p})\mathrm{d}t$$

$$\delta \int L\mathrm{d}t = 0 \quad \rightarrow \quad L - \dot{p}\frac{\partial L}{\partial \dot{p}} = \frac{\dot{p}^2}{(Cp+\dot{p})[C(1-p)-\dot{p}]} = K$$

$$p(t) = \frac{1}{\mathrm{e}^{\varepsilon(t)/kT}+1}\left[1+\sqrt{K\mathrm{e}^{\varepsilon(t)/kT}}\right]$$

K measures the degree of dissipation.

Carnot limit: $K \rightarrow 0$

Determination of K

$$C_\tau = F(p(\tau), K) - F(p(0), K)$$

$$F(p,K) = -\frac{1}{2}\ln p + \frac{1}{\sqrt{K}}\arctan\left[\frac{1-2p}{\sqrt{K+4p(1-p)}}\right] + \frac{1}{2}\ln\left[\frac{2p+K+\sqrt{K^2+4Kp(1-p)}}{2(1-p)+K+\sqrt{K^2+4Kp(1-p)}}\right]$$

Exact Entropy flow

$$\Delta S^{\mathrm{e}} = \mathcal{S}(p(\tau), K) - \mathcal{S}(p(0), K)$$

$$\mathcal{S}(p,K) = p\ln\left[\frac{2p(1-p)+K-\sqrt{K^2+4Kp(1-p)}}{2p^2}\right] - \sqrt{K}\arcsin\left[\frac{1-2p}{\sqrt{K+1}}\right]$$
$$-\ln\left[\frac{2(1-p)-K-\sqrt{K^2+4Kp(1-p)}}{2}\right]$$

Solution at Weak Dissipation Limit $K \ll 1$

For given τ_C, τ_H, p_0, p_1

Optimum protocol

$$p_\mathrm{C}(t) = \frac{1}{2}\left[1 - \sin\left(\frac{t}{\tau_\mathrm{C}}|\phi_1 - \phi_0| + \phi_0\right)\right]$$

$$p_\mathrm{H}(t) = \frac{1}{2}\left[1 - \sin\left(\frac{t}{\tau_\mathrm{C}}|\phi_1 - \phi_0| - \phi_0\right)\right]$$

$$\phi_i = \arcsin(1 - 2p_i) \quad i = 0,1$$

Entropy change

$$\Delta S_\mathrm{H} = \Delta S_\mathrm{rev} - \frac{(\phi_1 - \phi_0)^2}{C_\mathrm{H}}\frac{1}{\tau_\mathrm{H}} \qquad \Delta S_\mathrm{C} = -\Delta S_\mathrm{rev} - \frac{(\phi_1 - \phi_0)^2}{C_\mathrm{C}}\frac{1}{\tau_\mathrm{C}}$$

Maximum power

$$P = \frac{Q_\mathrm{H} + Q_\mathrm{C}}{\tau_\mathrm{H} + \tau_\mathrm{C}} = \frac{(T_\mathrm{H} - T_\mathrm{C})\Delta S_\mathrm{rev} - (\phi_1 - \phi_0)^2\left[\frac{T_\mathrm{H}}{C_\mathrm{H}\tau_\mathrm{H}} + \frac{T_\mathrm{C}}{C_\mathrm{C}\tau_\mathrm{C}}\right]}{\tau_\mathrm{H} + \tau_\mathrm{C}}$$

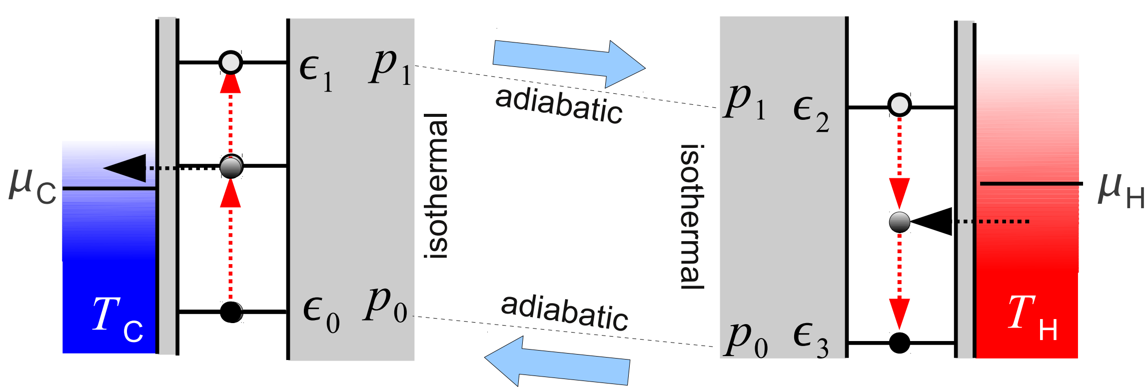
Further maximization w.r.t. τ_{H} and τ_{C}

$$\eta^* = 1 + \frac{Q_{\mathrm{C}}^*}{Q_{\mathrm{H}}^*} = \frac{\eta_{\mathrm{C}}}{2} + \frac{\eta_{\mathrm{C}}^2}{4\left(1+\sqrt{\frac{C_{\mathrm{H}}}{C_{\mathrm{C}}}}\right)} + \frac{\eta_{\mathrm{C}}^3}{8\left(1+\sqrt{\frac{C_{\mathrm{H}}}{C_{\mathrm{C}}}}\right)} + \cdots$$

$$\frac{\eta_{\mathrm{C}}}{2} \leq \eta^* \leq \frac{\eta_{\mathrm{C}}}{2-\eta_{\mathrm{C}}}$$

Lower bound: $\frac{C_{\mathrm{H}}}{C_{\mathrm{C}}} \to \infty$

Upper bound: $\frac{C_{\mathrm{H}}}{C_{\mathrm{C}}} \to 0$



Conclusions

1) The efficiency at maximum power is derived without a specific model at the weak dissipation limit, .

2) The efficiency at maximum power is bounded between $\eta_C/2$ and $\eta_C/(2-\eta_C)$

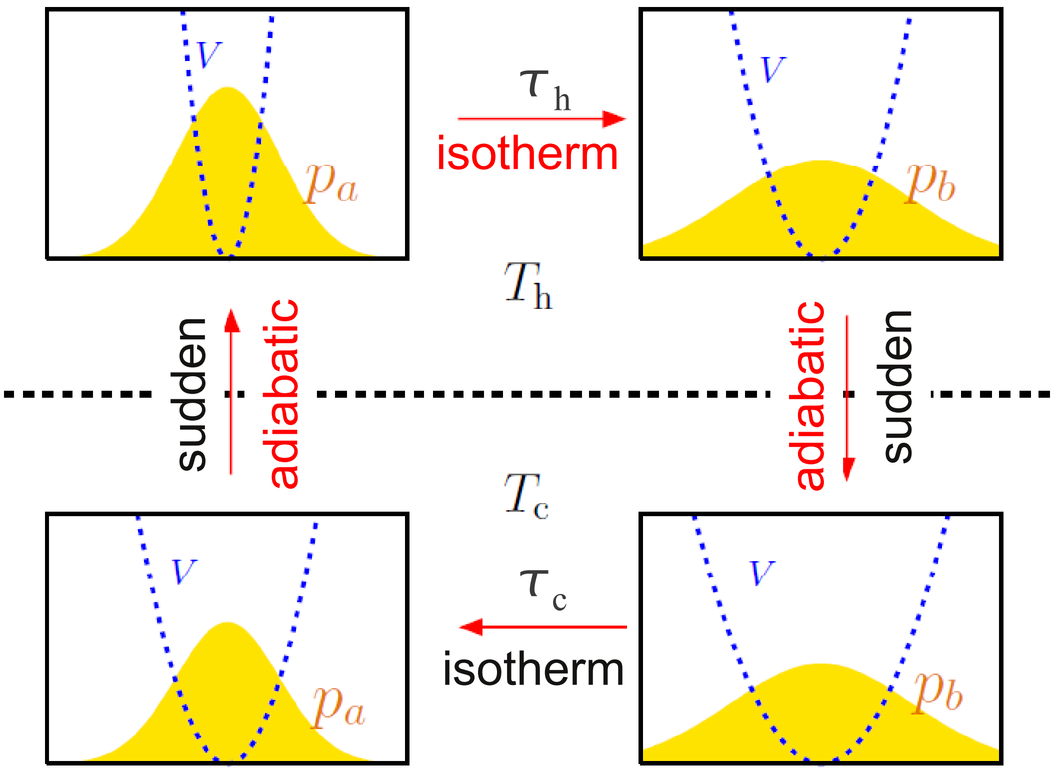
3) Exact Curzon-Ahlborn efficiency is obtained when left-right symmetry holds and it lies between the lower and upper bounds.

4) Only maximization of power with respect to operation times is necessary to get the Curzon-Ahlborn efficiency

5) The method of asymptotic expansion (weak dissipation limit) is justified for general Markovian processes.

6) The present results are demonstrated using analytically solvable model based on a quantum dot Carnot engine.

Brownian heat engine



Controle parameter: spring constant

Thermodynamic state: density *p*(*x*)

$$\eta^* = \frac{2(T_h - T_c)}{3T_h + T_c} = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + \frac{\eta_C^3}{32} + \dots$$

Schmiedl & Seifert (2008)